# MA 105 : Calculus
# Division 3, Lecture 29

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# Recap of the previous lecture

- Fubini Theorem over Elementary Regions. Examples
- Area of a bounded subset of $R^2$
- General version of domain additivity
- Change of Variables in a Double Integral. Example
- Polar transformation, Example
- Triple integrals

# Triple Integrals

The theory of double integrals of functions of two variables readily extends to triple integrals of functions of three variables. No new ideas are needed. We shall, therefore, mention only a few important points.

Consider a **cuboid** $K := [a, b] \times [c, d] \times [p, q]$, where $a < b$, $c < d$, $p < q$. We consider a partition $P$ of $K$ into several subcuboids.

For a bounded function $f : K \to \mathbb{R}$, we define the corresponding lower triple sum $L(P, f)$, the upper triple sum $U(P, f)$, the lower triple integral $L(f)$, and the upper triple integral $U(f)$. As before, $f$ is **(triple) integrable** on $K$, if $L(f) = U(f)$. In this case, its **triple integral** is denoted by

$$\iiint_K f(x, y, z) d(x, y, z) \quad \text{or} \quad \iiint_K f.$$

If $f := 1$ on $K$, then $f$ is integrable on $K$, and its triple integral is equal to the volume of $K$, whereas if $f$ is the trivariate Dirichlet function on $K$, then $f$ is not integrable on $K$.

Further, $f$ is integrable on $K$ if and only if it satisfies the Riemann condition. If $f$ is monotonic in each of the three variables, or if $f$ is continuous on $K$, then $f$ is integrable.

## Fubini's theorem on cuboids

Let $f$ be integrable on $K$, and let $I$ denote its triple integral. Suppose for each fixed $x \in [a,b]$, the double integral $\iint_{[c,d]\times[p,q]} f(x,y,z)d(y,z)$ exists. Then the **iterated integral** $\int_a^b \big(\iint_{[c,d]\times[p,q]} f(x,y,z)d(y,z)\big)dx$ exists and equals $I$. Further, if for each fixed $(x,y) \in [a,b]\times[c,d]$, the Riemann integral $\int_p^q f(x,y,z)dz$ exists, then the **iterated integral** $\int_a^b \big[\int_c^d \big(\int_p^q f(x,y,z)dz\big)dy\big]dx$ exists and equals $I$.

There are other versions of this result with the roles of $x, y, z$ interchanged. They are used in evaluating triple integrals.

Let $D$ be a bounded subset of $\mathbb{R}^3$, and let $f : D \to \mathbb{R}$ be a bounded function. Consider a cuboid $K := [a, b]\times[c, d]\times[p, q]$ such that $D \subset K$, and define $f^* : K \to \mathbb{R}$ by

$$f^*(x, y, z) := \begin{cases} f(x, y, z) & \text{if } (x, y, z) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

We say that $f$ is **integrable** over $D$ if $f^*$ is integrable on $K$, and in this case, the **triple integral** of $f$ (over $D$) is defined to be the triple integral of $f^*$ (on $K$), that is,

$$\iiint_D f(x, y, z)d(x, y, z) := \iiint_K f^*(x, y, z)d(x, y, z).$$

We may also denote the triple integral by $\iiint_D f$.

The triple integral over a bounded subset of $\mathbb{R}^3$ has algebraic and order properties analogous to those of a double integral over a bounded subset of $\mathbb{R}^2$.

In order to seek conditions under which a bounded function $f$ defined on a bounded subset $D$ of $\mathbb{R}^3$ is integrable over $D$, we introduce the following concept.

A bounded subset $E$ of $\mathbb{R}^3$ is of (three-dimensional) **content zero** if for every $\epsilon > 0$, there are finitely many cuboids whose union contains $E$ and the sum of whose volumes is less than $\epsilon$.

Examples:
(i) Let $D_0$ be a bounded subset of $\mathbb{R}^2$, and let $\varphi : D_0 \to \mathbb{R}$ be integrable over $D_0$. Then its graph $E := \{(x, y, \varphi(x, y)) : (x, y) \in D_0\}$ is of (three-dimensional) content zero. To see this, let $R$ be a rectangle containing $D_0$, and let $\varphi^* : R \to \mathbb{R}$ be the usual extension of $\varphi$. Let $\epsilon > 0$. By the Riemann condition for the integrable function $\varphi^*$, there is a partition $P := \{(x_i, y_j) : i = 0, 1, \ldots, n \text{ and } j = 0, 1, \ldots, k\}$ of $R$ such that $U(P, \varphi^*) - L(P, \varphi^*) < \epsilon$. Then the $nk$ cuboids $K_{i,j} := [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [m_{i,j}(\varphi^*), M_{i,j}(\varphi^*)], i = 1, \ldots n; j = 1, \ldots, k$, work.

(ii) Let functions $x$, $y$, $z : [a,b]\times[c,d] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)\times(c,d)$, and let their partial derivatives be bounded on $(a,b)\times(c,d)$. Then the set $E := \{(x(u,v), y(u,v), z(u,v)) \in \mathbb{R}^3 : (u,v) \in [a,b]\times[c,d]\}$ is of three-dimensional content zero. [We shall omit the proof (which involves he Bivariate Mean Value Theorem.]

The above result can be improved by replacing the rectangle $[a,b]\times[c,d]$ by a bounded convex subset $E$ of $\mathbb{R}^2$ whose interior $E^\circ$ is nonempty.

Further, if $E$ is any bounded convex subset of $\mathbb{R}^2$, and $x$, $y$ and $z$ are differentiable functions on an open subset of $\mathbb{R}^2$ containing $E$, and if their first partial derivatives are bounded on $E$, then we can prove that the set

$$S := \{(x(u,v), y(u,v), z(u,v)) \in \mathbb{R}^3 : (u,v) \in E\}$$

is of three-dimensional content zero.

Remark: The subset $E := [a, b] \times [c, d] \times \{0\}$ of $\mathbb{R}^3$ is of three-dimensional content zero, but the subset $[a, b] \times [c, d]$ of $\mathbb{R}^2$ is not of two-dimensional content zero.

**Theorem**

Let $D$ be a bounded subset of $\mathbb{R}^3$, and $f : D \to \mathbb{R}$ be a bounded function. Suppose (i) the set of discontinuities of $f$ in $D$ is of (three-dimensional) content zero and (ii) the boundary $\partial D$ of $D$ is of (three-dimensional) content zero. Then $f$ is integrable over $D$.

We omit the proof.

Suppose $D_0$ is a subset of $\mathbb{R}^2$ having an area, that is, $\partial D_0$ is of two-dimensional content zero. Let $\psi_1, \psi_2 : D_0 \to \mathbb{R}$ be continuous, and let $\psi_1 \leq \psi_2$. Consider an **elementary region**

$$D := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D_0 \text{ and } \psi_1(x, y) \leq z \leq \psi_2(x, y)\}.$$

Then $\partial D$ is of three-dimensional content zero. Hence if a function is continuous on $D$, then it is integrable over $D$.

## Proposition (Cavalieri Principle)

Let $D$ be a bounded subset of $\mathbb{R}^3$, and let $f : D \to \mathbb{R}$ be continuous. Suppose $D := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D_0$ and $\psi_1(x, y) \leq z \leq \psi_2(x, y)\}$, where $D_0$ is a subset of $\mathbb{R}^2$ having an area, $\psi_1, \psi_2 : D_0 \to \mathbb{R}$ are continuous and $\psi_1 \leq \psi_2$. Then $f$ is integrable on $D$, the **iterated integral** exists, and

$$\iiint_D f = \iint_{D_0} \left( \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z) dz \right) d(x, y).$$

In particular, if $D_0$ is an elementary region in $\mathbb{R}^2$ given by $D_0 := \{(x, y) : \mathbb{R}^2 : a \leq x \leq b$ and $\phi_1(x) \leq y \leq \phi_2(x)\}$, where $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ are continuous and $\phi_1 \leq \phi_2$, then

$$\iiint_D f = \int_a^b \left( \int_{\phi_1(x)}^{\phi_2(x)} \left( \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z)\, dz \right) dy \right) dx.$$

The Cavalieri Principle stated above gives a version of Fubini's theorem for an 'elementary domain' $D$ in $\mathbb{R}^3$. Other versions can be obtained similarly.

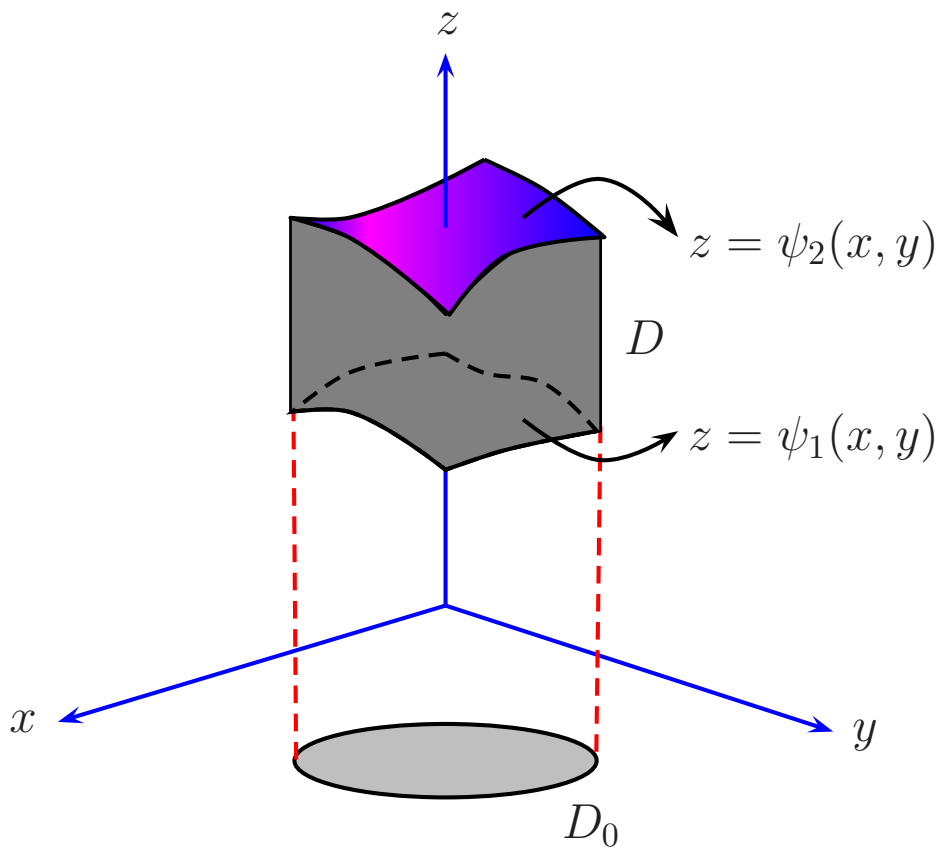



Figure: Illustration of the Cavalieri Principle: a solid between two surfaces defined over $D_0$.

# Constant Function on a Bounded Subset of $\mathbb{R}^3$

Let $D$ be a bounded subset of $\mathbb{R}^3$. Define

$$1_D : D \to \mathbb{R} \quad \text{by} \quad 1_D(x, y, z) := 1 \text{ for all } (x, y, z) \in D.$$

If the boundary $\partial D$ of $D$ is of (three-dimensional) content zero, then the continuous function $1_D$ is integrable on $D$. The converse also holds, that is, if the function $1_D$ is integrable on $D$, then $\partial D$ is of (three-dimensional) content zero.

Examples:
Let $K := [a, b] \times [c, d] \times [p, q]$. If $D := K$, then $1_D$ is integrable and its triple integral is equal to $(b - a)(d - c)(q - p)$. But if $D := \{(x, y, z) \in K : x, y, z \in \mathbb{Q}\}$, then the function $1_D^* : K \to \mathbb{R}$, obtained by extending the function $1_D$ as usual, is the trivariate Dirichlet function on $K$. Now $1_D^*$ is not integrable on $K$, that is, $1_D$ is not integrable over $D$. Note: In this case, $\partial D = K$ is not of three-dimensional content zero.

# Volume of a Bounded Subset of $\mathbb{R}^3$

Let $D \subset \mathbb{R}^3$ be bounded. We say that $D$ has a **volume** if the function $1_D$ is integrable over $D$, and then we define

$$\mathrm{Vol}\,(D) := \iiint_D 1_D(x,y,z)\,d(x,y,z).$$

Important special case: Consider an elementary domain $D := \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D_0$ and $\psi_1(x,y) \le z \le \psi_2(x,y)\}$, where $D_0$ is a subset of $\mathbb{R}^2$ having an area, $\psi_1, \psi_2 : D_0 \to \mathbb{R}$ are continuous and $\psi_1 \le \psi_2$.

By the Cavalier Principle, the volume of $D$ is equal to

$$\iint_{D_0} \left( \int_{\psi_1(x,y)}^{\psi_2(x,y)} dz \right) d(x,y) = \iint_{D_0} \big(\psi_2(x,y) - \psi_1(x,y)\big)\, d(x,y),$$

which was our definition (Lecture 27) of the **volume between the surfaces** $z = \psi_1(x,y)$ and $z = \psi_2(x,y)$, $(x,y) \in D_0$.

For instance, let

$$D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + 4z^2 \leq 6\}.$$

Then $D = \left\{(x, y, z) \in \mathbb{R}^3 : |x| \leq \sqrt{6},\ |y| \leq \sqrt{6 - x^2}\right.$ and $\left.|z| \leq \sqrt{6 - x^2 - y^2}/2\right\}$. Hence the volume of the ellipsoidal solid $D$ is equal to

$$8 \int_0^{\sqrt{6}} \left( \int_0^{\sqrt{6-x^2}} \left( \int_0^{\sqrt{6-x^2-y^2}/2} dz \right) dy \right) dx = 4\sqrt{6}\,\pi.$$

Note: It can be shown that the definitions of the volume of a solid used in the slice method, the washer method and the shell method agree with the general definition of the volume of a bounded subset of $\mathbb{R}^3$ given above.

The following basic inequality is important. Let $D$ be a bounded subset of $\mathbb{R}^3$ which has a volume. If $f : D \to \mathbb{R}$ is an integrable function, and if $|f| \leq \alpha$ on $D$, then

$$\left| \iiint_D f(x,y,z)d(x,y,z) \right| \leq \iiint_D |f(x,y,z)|d(x,y,z) \leq \alpha \,\mathrm{Vol}\,(D).$$

The following result is useful in evaluating a triple integral.

**Theorem (Domain Additivity)**

Let $D$ be a bounded subset of $\mathbb{R}^3$, and let $f : D \to \mathbb{R}$ be a bounded function. Suppose $D = D_1 \cup D_2$, where $D_1 \cap D_2$ is of (three-dimensional) content zero. If $f$ is integrable over $D_1$ and over $D_2$, then $f$ is integrable over $D$ and

$$\iiint_D f = \iiint_{D_1} f + \iiint_{D_2} f.$$

# Change of Variables in a Triple Integral

To simplify the calculation of $\iiint_D f(x,y,z)d(x,y,z)$, we may reduce it to another triple integral $\iiint_E g(u,v,w)d(u,v,w)$.

Let $E \subset \mathbb{R}^3$ and $\Phi := (\phi_1, \phi_2, \phi_3) : E \to \mathbb{R}^3$. Suppose $Q_0 := (u_0, v_0, w_0)$ is an interior point of $E$.
If $x := \phi_1(u,v,w)$, $y := \phi_2(u,v,w)$ and $z := \phi_3(u,v,w)$ for $(u,v,w) \in E$, and the functions $x$, $y$ and $z$ have partial derivatives at $Q_0$, then the **Jacobian** of $\Phi$ at $Q_0$ is given by

$$J(\Phi)(Q_0) := \frac{\partial(x,y,z)}{\partial(u,v,w)}(Q_0) := \det \begin{bmatrix} \frac{\partial x}{\partial u}(Q_0) & \frac{\partial x}{\partial v}(Q_0) & \frac{\partial x}{\partial w}(Q_0) \\ \frac{\partial y}{\partial u}(Q_0) & \frac{\partial y}{\partial v}(Q_0) & \frac{\partial y}{\partial w}(Q_0) \\ \frac{\partial z}{\partial u}(Q_0) & \frac{\partial z}{\partial v}(Q_0) & \frac{\partial z}{\partial w}(Q_0) \end{bmatrix}.$$

## Theorem (Change of Variables Formula)

Let $D$ be a closed and bounded subset of $\mathbb{R}^3$ which has a volume. Let $f : D \to \mathbb{R}$ be continuous. Suppose $\Omega$ is an open subset of $\mathbb{R}^3$ and $\Phi : \Omega \to \mathbb{R}^3$ is a one-one transformation such that $\Phi := (\phi_1, \phi_2, \phi_3)$, where $\phi_1, \phi_2, \phi_3$ have continuous partial derivatives in $\Omega$ and $J(\Phi)(u, v, w) \neq 0$ for all $(u, v, w) \in \Omega$. Let $E \subset \Omega$ be such that $\Phi(E) = D$. Then $E$ is a closed and bounded subset of $\Omega$, and $E$ has a volume. Moreover, $f \circ \Phi : E \to \mathbb{R}$ is continuous, and

$$\iiint_D f = \iiint_E (f \circ \Phi)(u, v, w) |J(\Phi)(u, v, w)| d(u, v, w).$$

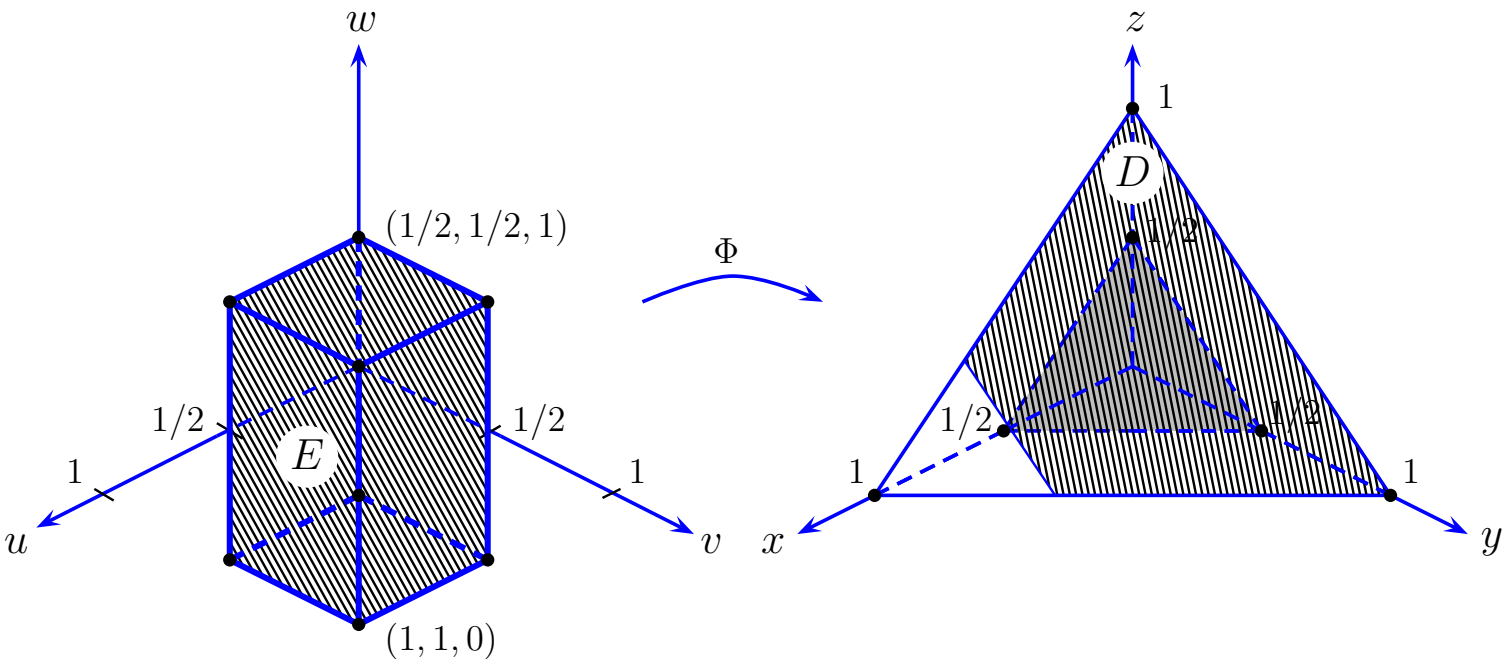
Note: The hypothesis $J(\Phi)(u, v, w) \neq 0$ for all $(u, v, w) \in \Omega$ can be weakened by assuming only that $J(\Phi)(u, v, w) \geq 0$ for all $(u, v, w) \in \Omega$ or $J(\Phi)(u, v, w) \leq 0$ for all $(u, v, w) \in \Omega$, and $J(\Phi)(u, v, w) = 0$ only for $(u, v, w)$ in a subset of $\Omega$ of (three-dimensional) content zero.

Example: Let

$D := \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0,\ x \leq y+z,\ 1 \leq 2(x+y+z) \leq 2\}$,

and define $f : D \to \mathbb{R}$ by $f(x, y, z) := z/(y + z)$.

(Note: $(x, y, z) \in D \implies 0 \leq x \leq x/2 + (y + z)/2 \leq 1/2$.)




Let $u := x + y + z$, $\quad v := \dfrac{y + z}{x + y + z}$, $\quad w := \dfrac{z}{y + z}$,

that is, $x := u(1 - v)$, $\quad y := u\,v(1 - w)$, $\quad z := u\,v\,w$.

Consider $\Omega := \{(u, v, w) \in \mathbb{R}^3 : u > 0 \text{ and } v > 0\}$, and define $\Phi : \Omega \to \mathbb{R}^3$ by $\Phi(u, v, w) := \big(u(1-v), u\,v(1-w), u\,v\,w\big)$. Now $\Phi$ maps $\Omega$ onto $\{(x, y, z) \in \mathbb{R}^2 : x + y + z > 0 \text{ and } y + z > 0\}$ and is one-one on $\Omega$. Also, if $\Phi = (\phi_1, \phi_2, \phi_3)$, then the partials of $\phi_1$, $\phi_2$, and $\phi_3$ are continuous, and for all $(u, v, w) \in \Omega$,

$$J(\Phi)(u, v, w) = \det \begin{bmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -u\,v \\ v\,w & u\,w & u\,v \end{bmatrix} = u^2 v \neq 0.$$

Let $E := [1/2, 1] \times [1/2, 1] \times [0, 1]$. Then $\Phi(E) = D$, and $(f \circ \Phi)(u, v, w) = w$ for $(u, v, w) \in E$. By the CoV formula,

$$\begin{aligned} \iiint_D f &= \iiint_E f(u(1-v), u\,v(1-w), u\,v\,w)\,|u^2 v|\,d(u, v, w) \\ &= \iiint_E u^2 v\,w\,d(u, v, w) \\ &= \left(\int_{1/2}^1 u^2\,du\right)\left(\int_{1/2}^1 v\,dv\right)\left(\int_0^1 w\,dw\right) = \frac{7}{128}. \end{aligned}$$

Two important cases involving a change of variables in triple integrals are given by switching to cylindrical coordinates, and to spherical coordinates.

(i) Cylindrical coordinates: Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\Phi(r, \theta, z) := (r\cos\theta,\ r\sin\theta,\ z) \quad \text{for } (r, \theta, z) \in \mathbb{R}^3.$$

Then for all $(r, \theta, z) \in \mathbb{R}^3$,

$$J(\Phi)(r, \theta, z) = \det \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = r.$$

The Jacobian of $\Phi$ is nonzero on $\{(r, \theta, z) \in \mathbb{R}^3 : r \neq 0\}$.

The function $\Phi : \{(r, \theta, z) \in \mathbb{R}^3 : r > 0 \text{ and } -\pi < \theta \leq \pi\} \to \{(x, y, z) \in \mathbb{R}^3 : (x, y) \neq (0, 0)\}$ is one-one and onto.

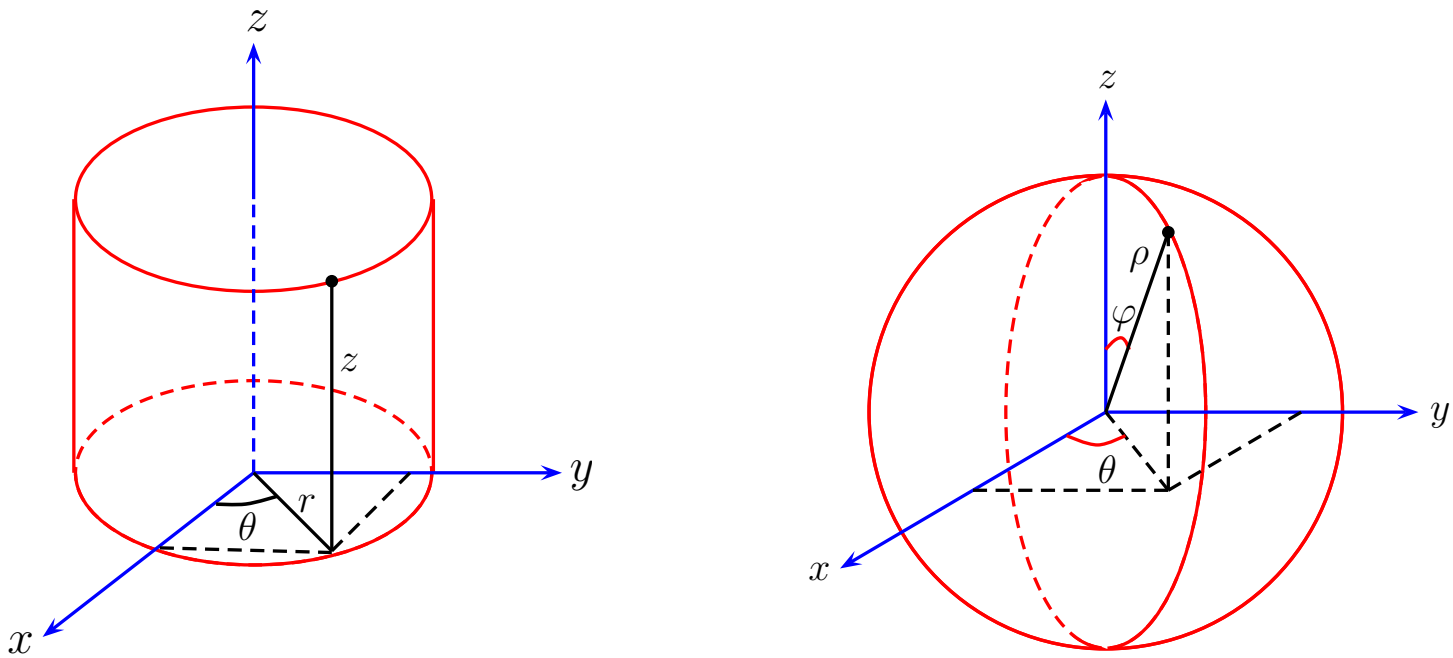



Figure: Illustrations of cylindrical and spherical coordinates.

(ii) Spherical coordinates: Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\Phi(\rho, \varphi, \theta) := (\rho \sin\varphi \cos\theta,\ \rho \sin\varphi \sin\theta,\ \rho \cos\varphi) \quad \text{for } (\rho, \varphi, \theta) \in \mathbb{R}^3.$$

Then for all $(\rho, \varphi, \theta) \in \mathbb{R}^3$,

$$\begin{aligned} J(\Phi)(\rho, \varphi, \theta) &= \det \begin{bmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{bmatrix} \\ &= \rho^2 \sin\varphi. \end{aligned}$$

The Jacobian of $\Phi$ is nonzero on the set
$\{(\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho \neq 0$ and $\varphi \neq m\pi$ for any $m \in \mathbb{Z}\}$.

The function $\Phi : \{(\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho > 0,\ 0 < \varphi < \pi$ and $-\pi < \theta \leq \pi\} \to \{(x, y, z) \in \mathbb{R}^3 : (x, y) \neq (0, 0)\}$ is one-one and onto.

In these cases the following results hold.

## Proposition

Let $D$ be a closed and bounded subset of $\mathbb{R}^3$ which has a volume, and let $f : D \to \mathbb{R}$ be continuous.

(i) If $E := \{(r, \theta, z) \in \mathbb{R}^3 : r \geq 0, -\pi \leq \theta \leq \pi$ and $(r\cos\theta,\ r\sin\theta,\ z) \in D\}$, and if $E$ has a volume, then the triple integral of $f$ over $D$ is equal to

$$\iiint_E f(r\cos\theta, r\sin\theta, z)\, r\, d(r, \theta, z).$$

(ii) If $E := \{(\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho \geq 0,\ 0 \leq \varphi \leq \pi,\ -\pi \leq \theta \leq \pi$ and $(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) \in D\}$, and if $E$ has a volume, then the triple integral of $f$ over $D$ is equal to

$$\iiint_E f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, \rho^2 \sin\varphi\ d(\rho, \varphi, \theta).$$

Examples

(i) Let $D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1 \text{ and } 0 \leq z \leq 1\}$, and define $f : D \to \mathbb{R}$ by $f(x, y, z) := z\sqrt{1 - x^2 - y^2}$. Then

$$\begin{aligned} E &:= \{(r, \theta, z) \in \mathbb{R}^3 : r \geq 0,\ -\pi \leq \theta \leq \pi,\ (r\cos\theta, r\sin\theta, z) \in D\} \\ &= [0, 1] \times [-\pi, \pi] \times [0, 1]. \end{aligned}$$

Hence

$$\begin{aligned} \iiint_D f &= \iiint_E f(r\cos\theta, r\sin\theta, z)\, r\, d(r, \theta, z) \\ &= \int_0^1 \left[ \int_{-\pi}^{\pi} \left( \int_0^1 z\sqrt{1 - r^2}\, r\, dz \right) d\theta \right] dr \\ &= \pi \int_0^1 \sqrt{1 - r^2}\, r\, dr = \frac{\pi}{2} \int_0^1 \sqrt{s}\, ds = \frac{\pi}{3}. \end{aligned}$$

(ii) Let $a > 0$ and $D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq a^2\}$. Define $f : D \to \mathbb{R}$ by $f(x, y, z) = z^2$. Then

$$\begin{aligned} E &:= \{(\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho \geq 0,\ 0 \leq \varphi \leq \pi,\ -\pi \leq \theta \leq \pi \text{ and} \\ &\qquad (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi) \in D\} \\ &= [0, a] \times [0, \pi] \times [-\pi, \pi]. \end{aligned}$$

Hence the triple integral of $f$ over $D$ is equal to

$$\begin{aligned} &\iiint_E f(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)\, \rho^2 \sin\varphi \, d(\rho, \varphi, \theta) \\ &= \int_0^a \left[ \int_0^\pi \left( \int_{-\pi}^\pi (\rho^2 \cos^2\varphi)\, \rho^2 \sin\varphi \, d\theta \right) d\varphi \right] d\rho \\ &= 2\pi \int_0^a \rho^4 \left( \int_0^\pi \cos^2\varphi \sin\varphi \, d\varphi \right) d\rho = \frac{2\pi a^5}{5} \cdot \frac{2}{3} = \frac{4\pi a^5}{15}. \end{aligned}$$

# Vector Algebra

We begin a part of this course known as '**Vector Analysis**'.
It deals with scalar fields and vector fields.
Initially, we recall some concepts in '**Vector Algebra**'.
In Lecture 20, we have defined the **Euclidean space**

$$\mathbb{R}^m := \{\boldsymbol{x} = (x_1, \ldots, x_m) : x_i \in \mathbb{R} \text{ for } i = 1, \ldots, m\},$$

where $m \in \mathbb{N}$. An element of $\mathbb{R}^1 := \mathbb{R}$ is called a **scalar**, and an element of $\mathbb{R}^m$ is called a **vector** if $m \geq 2$.

Let $\boldsymbol{x} := (x_1, \ldots, x_m)$, $\boldsymbol{y} := (y_1, \ldots, y_m) \in \mathbb{R}^m$, and $a \in \mathbb{R}$.
We have already defined the sum $\boldsymbol{x} + \boldsymbol{y}$ and the scalar multiple $a\boldsymbol{x}$. Also, we have studied the **dot product** or the **scalar product** of $\boldsymbol{x}$ and $\boldsymbol{y}$ defined by

$$\boldsymbol{x} \cdot \boldsymbol{y} := x_1 y_1 + \cdots + x_m y_m \in \mathbb{R}.$$

# Vector Product

Let $m := 3$. Consider $\boldsymbol{x} := (x_1, x_2, x_3)$ and $\boldsymbol{y} := (y_1, y_2, y_3)$. The **cross product** or the **vector product** of $\boldsymbol{x}$ with $\boldsymbol{y}$ is defined by

$$\boldsymbol{x} \times \boldsymbol{y} := (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1) \in \mathbb{R}^3.$$

$$\text{Let} \quad \boldsymbol{i} := (1,0,0), \quad \boldsymbol{j} := (0,1,0), \quad \boldsymbol{k} := (0,0,1).$$

Then $\boldsymbol{x} := x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$, $\boldsymbol{y} := y_1\boldsymbol{i} + y_2\boldsymbol{j} + y_3\boldsymbol{k}$, and

$$\begin{aligned} \boldsymbol{x} \times \boldsymbol{y} &= (x_2y_3 - x_3y_2)\boldsymbol{i} + (x_3y_1 - x_1y_3)\boldsymbol{j} + (x_1y_2 - x_2y_1)\boldsymbol{k} \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}. \end{aligned}$$

Note: $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}$, $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$, $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$, $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$.

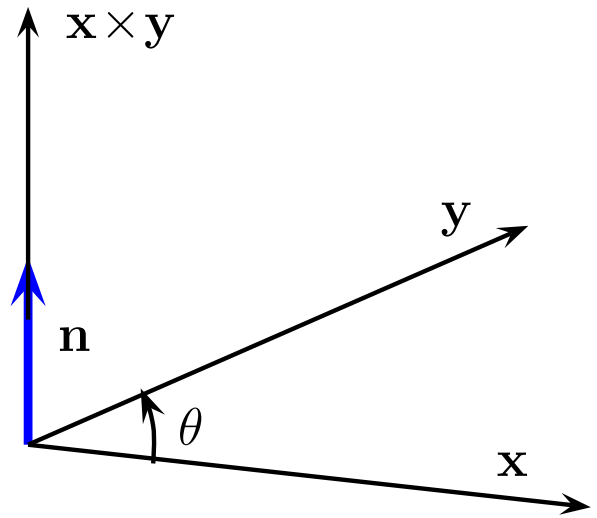
Properties of determinants show that for every $\boldsymbol{z} \in \mathbb{R}^3$,

$$(\boldsymbol{x}+\boldsymbol{y}) \times \boldsymbol{z} = (\boldsymbol{x} \times \boldsymbol{z}) + (\boldsymbol{y} \times \boldsymbol{z}) \quad \text{and} \quad \boldsymbol{y} \times \boldsymbol{x} = -(\boldsymbol{x} \times \boldsymbol{y}).$$

Let $\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{y} \neq \boldsymbol{0}$. It can be shown that

$$\boldsymbol{x} \times \boldsymbol{y} = \|\boldsymbol{x}\| \, \|\boldsymbol{y}\| (\sin\theta)\, \boldsymbol{n},$$

where $\theta \in [0, \pi]$ is the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$, and $\boldsymbol{n}$ is the unit vector which is perpendicular to the plane containing $\boldsymbol{x}$ and $\boldsymbol{y}$, and obeys the 'right-hand rule'.




Hence $\|\boldsymbol{x} \times \boldsymbol{y}\|$ = the area of the parallelogram with sides $\boldsymbol{x}$, $\boldsymbol{y}$.

# Scalar Triple Product and Vector triple Product

Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^3$. Then $\boldsymbol{x} \cdot (\boldsymbol{y} \times \boldsymbol{z}) \in \mathbb{R}$ and $\boldsymbol{x} \times (\boldsymbol{y} \times \boldsymbol{z}) \in \mathbb{R}^3$ are called the **scalar triple product** and the **vector triple product** of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ respectively. It is easy to see that if $\boldsymbol{x} := (x_1, x_2, x_3)$, $\boldsymbol{y} := (y_1, y_2, y_3)$, $\boldsymbol{z} := (z_1, z_2, z_3)$, then

$$\boldsymbol{x} \cdot (\boldsymbol{y} \times \boldsymbol{z}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

Geometrically, $\boldsymbol{x} \cdot (\boldsymbol{y} \times \boldsymbol{z})$ can be interpreted as the (signed) volume of the parallelopiped defined by the vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$.

One can prove the **Lagrange formula**

$$\boldsymbol{x} \times (\boldsymbol{y} \times \boldsymbol{z}) = (\boldsymbol{x} \cdot \boldsymbol{z})\, \boldsymbol{y} - (\boldsymbol{x} \cdot \boldsymbol{y})\, \boldsymbol{z}$$

by considering each component of the LHS and the RHS.