MA 105 : Calculus
Division 3, Lecture 03

Prof. Sudhir R. Ghorpade
IIT Bombay

Recap of the previous lecture

- Intervals: open, closed, semi-open, infinite
- Functions. Notions of one-one, onto, and bijective functions; Inverse functions
- Examples of functions. Mention of:
  - Polynomials functions
  - Rational functions
  - Algebraic functions
  - Transcendental functions.
  - Functions obtained by piecing together known functions
- Sequences: Basic definitions and examples
- Boundedness. Examples of bounded and unbounded sequences
- Notion of convergence of a sequence. A few examples

Definition of convergence of a sequence

Definition

Let (a_n) be a sequence of real numbers. We say that (a_n) is convergent if there is $a \in \mathbb{R}$ such that the following condition holds. For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$|a_n - a| < \epsilon \text{ for all } n \geq n_0.$$

In this case, we say that (a_n) **converges** to a, or that a is a **limit** of (a_n), and we write $\lim_{n\to\infty} a_n = a$ or $a_n \to a$ as $n \to \infty$).

The part in red may be stated as: $|a_n - a| < \epsilon$ for all large n. In general, when we say that a statement holds for all large n, it means that there is $n_0 \in \mathbb{N}$ such that it holds for all $n \geq n_0$.

If a sequence is not convergent, then it is said to be **divergent**, *or we say that the sequence* **diverges**.

The convergence of a sequence is unaltered if a finite number of its terms are replaced by some other terms.

Examples:

(i) Let $a \in \mathbb{R}$ and $a_n := a$ for all $n \in \mathbb{N}$. Then $a_n \to a$. We can let $n_0 := 1$.

(ii) $a_n := 1/n$ for all $n \in \mathbb{N}$. Then $a_n \to 0$.

Let $\epsilon > 0$ be given. We want to find $n_0 \in \mathbb{N}$ such that $|(1/n) - 0| < \epsilon$ for all $n \geq n_0$.

Choose any $n_0 \in \mathbb{N}$ which is greater than $1/\epsilon$. This is possible because of the Archimedean property of $\mathbb{R}$. For example, we can let $n_0 := [1/\epsilon] + 1$.

(iii) $a_n := 2/(n^2+1)$ for $n \in \mathbb{N}$. Then $a_n \to 0$. Now

$$\left|\frac{2}{n^2+1} - 0\right| = \frac{2}{n^2+1} < \frac{2}{n^2} \quad \text{for all } n \in \mathbb{N}.$$

Choose $n_0 \in \mathbb{N}$ such that $n_0 > \sqrt{2}/\sqrt{\epsilon}$. For example, let $n_0 := [\sqrt{2}/\sqrt{\epsilon}] + 1$. Then $|a_n - 0| < \epsilon$ for all $n \geq n_0$.

(iv) $a_n := 5/(3n+1)$ for $n \in \mathbb{N}$. Then $a_n \to 0$. Now

$$\frac{5}{3n+1} < \frac{5}{3n} \quad \text{for all } n \in \mathbb{N}.$$

Choose $n_0 \in \mathbb{N}$ such that $n_0 > 5/3\epsilon$. For example, let $n_0 := [5/3\epsilon] + 1$. Then $|a_n - 0| < \epsilon$ for all $n \geq n_0$.

(v) $a_n := (-1)^n$ for all $n \in \mathbb{N}$. Then the sequence (a_n) is divergent, that is, it is not convergent.

Suppose (a_n) is convergent. Then there is a real number a such that $a_n \to a$. Let $\epsilon := 1/2$. Find $n_0 \in \mathbb{N}$ such that

$$|(-1)^n - a| < \frac{1}{2} \text{ for all } n \geq n_0.$$

Since $(-1)^{2n_0} = 1$ and $(-1)^{2n_0+1} = -1$,

$$\begin{aligned} 2 &= |(-1)^{2n_0} - (-1)^{2n_0+1}| \\ &\leq |(-1)^{2n_0} - a| + |a - (-1)^{2n_0+1}| \\ &< \frac{1}{2} + \frac{1}{2} = 1, \end{aligned}$$

which is a contradiction.

Uniqueness of limit

Theorem

A convergent sequence has a unique limit.

Proof: Let (a_n) be a sequence. Assume for a moment that $a_n \to a$ and $a_n \to b$, where $a \neq b$. Let $\epsilon := |a-b|/2 > 0$.

Let $n_0 \in \mathbb{N}$ be such that $n \geq n_0 \implies |a_n - a| < \epsilon$, and let $m_0 \in \mathbb{N}$ be such that $n \geq m_0 \implies |a_n - b| < \epsilon$. Consider $n := \max\{n_0, m_0\}$. Then

$$|a-b| \leq |a-a_n| + |a_n - b| < \epsilon + \epsilon = |a-b|,$$

which is a contradiction. Hence $a = b$. $\square$

A useful result

Theorem

Every convergent sequence is bounded.

Proof: Suppose $a_n \to a$. Let $\epsilon := 1$. There is $n_0 \in \mathbb{N}$ such that

$$|a_n - a| < 1 \text{ for all } n > n_0.$$

Hence

$$|a_n| \leq |a_n - a| + |a| < 1 + |a| \text{ for all } n > n_0.$$

Define $\alpha := \max\{|a_1|, \ldots, |a_{n_0}|, |a| + 1\}$. Then $|a_n| \leq \alpha$ for all $n \in \mathbb{N}$. Hence (a_n) is bounded. □

- A bounded sequence need not be convergent.
  Example: $a_n := (-1)^n$ for $n \in \mathbb{N}$.
- If $a_n := (-1)^n n$ for $n \in \mathbb{N}$, then (a_n) is divergent since it is not bounded.

Limit theorems for sequences

Suppose (a_n) and (b_n) are convergent sequences. Then

- $\lim_{n\to\infty}(a_n \pm b_n) = (\lim_{n\to\infty} a_n) \pm (\lim_{n\to\infty} b_n)$ resp.,
- $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.

  In particular, $\lim_{n\to\infty}(r\, a_n) = r(\lim_{n\to\infty} a_n)$ for $r \in \mathbb{R}$.
- If $\lim_{n\to\infty} b_n \neq 0$, then there is $\delta > 0$ such that $|b_n| \geq \delta > 0$ for all large n and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

- If $a_n \leq b_n$ for all large $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.
- (Sandwich theorem) If $a_n \leq c_n \leq b_n$ for all large $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = c = \lim_{n\to\infty} b_n$, then $\lim_{n\to\infty} c_n = c$.

Examples

(i) Let $a \in \mathbb{R}$, and $a_n := a^n$ for $n \in \mathbb{N}$.
Then (a_n) is convergent $\iff$ $-1 < a \leq 1$.

Clearly, if $a := 0$, then $a_n \to 0$, and if $a := 1$, then $a_n \to 1$.
Also, if $a := -1$, then we have seen that (a_n) is divergent.

Let $0 < |a| < 1$, and $r := 1/|a|$. Then $r > 1$, and so $r = 1 + h$ with $h > 0$. By the binomial theorem,

$$r^n = (1 + h)^n = 1 + nh + \cdots + h^n > nh \quad \text{for all } n \in \mathbb{N}.$$

Hence $0 \leq |a_n| = |a|^n = (1/r^n) \leq (1/nh) \to 0$. Thus $a_n \to 0$.

Let $s := |a| > 1$. Then $s = 1 + h$ with $h > 0$, and for all $n \in \mathbb{N}$, $|a_n| = s^n > nh$. Hence (a_n) is unbounded, and so it is divergent.

Examples (continued)

(ii) Let $a_n := \dfrac{n^3 + 3n^2 + 1}{n^4 + 8n^2 + 2}$ for $n \in \mathbb{N}$. Then $a_n \to 0$,

since $0 \leq a_n \leq \dfrac{1}{n} + \dfrac{3}{8n^2} + \dfrac{1}{n^4} \to 0.$

(iii) Let $a_n := \dfrac{1}{n} \sin\left(\dfrac{1}{n}\right)$ for $n \in \mathbb{N}$. Then $a_n \to 0$,

since $|a_n| \leq \dfrac{1}{n} \to 0.$