MA 105 : Calculus
Division 1, Lecture 32

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Recap of the previous lecture

- Line integrals of scalar fields
- Reparametrization
- Invariance of line integrals of scalar fields under reparametrizatons
- Arc length parametrization of a regular path. Examples
- Line integrals of scalar fields along geometric curves
- Line integrals of vector fields
- Invariance of line integrals of vector fields under reparametrizaton up to a sign
- Line integrals as work, flow and circulation
- Flux integrals along simple closed regular paths in 2-space
- Preliminary discussion about path independence of line integrals

Path Independence of Line Integrals

Let $m \in \mathbb{N}$, $D \subset \mathbb{R}^m$, and consider a path $\gamma : [\alpha, \beta] \to \mathbb{R}^m$. We say that γ **lies in** D if $\gamma([\alpha, \beta]) \subset D$. Also, we say that $\gamma(\alpha)$ is the **initial point** and $\gamma(\beta)$ is the **final point** of γ.

Let $\boldsymbol{F} : D \to \mathbb{R}^m$ be a continuous vector field. We say that line integrals of $\boldsymbol{F}$ are **path-independent** in D if

$$\int_{\gamma} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\widetilde{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s}$$

for all (piecewise smooth) paths γ and $\widetilde{\gamma}$ lying in D which have the same initial point as well as the same final point.

Note: The path independence of line integrals of $\boldsymbol{F}$ in D does not mean that $\int_{\gamma} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\widetilde{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s}$ for any smooth paths γ and $\widetilde{\gamma}$ in D. It only says that this equality holds if the smooth paths γ and $\widetilde{\gamma}$ lie in D, and both have the same initial point, and both have the same final point.

Examples:

(i) Let $\boldsymbol{F}(x,y) := (x+y, x)$ for $(x,y) \in \mathbb{R}^2$. Consider a smooth path $\gamma : [\alpha, \beta] \to \mathbb{R}^2$. If $\gamma(t) := (x(t), y(t))$ for $t \in [\alpha, \beta]$, then

$$\begin{aligned}
\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_\alpha^\beta \big(x(t)+y(t), x(t)\big) \cdot (x'(t), y'(t)) dt \\
&= \int_\alpha^\beta \big(x(t)x'(t) + (xy)'(t)\big)\, dt \\
&= \frac{1}{2}x^2(\beta) - \frac{1}{2}x^2(\alpha) + x(\beta)y(\beta) - x(\alpha)y(\alpha).
\end{aligned}$$

Thus if smooth paths γ and $\widetilde{\gamma}$ have the same initial point, and also the same final point, then $\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\widetilde{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s}$. Hence line integrals of $\boldsymbol{F}$ are path-independent in $\mathbb{R}^2$.

(ii) Let $\boldsymbol{F}(x,y) := (x-y, x)$ for $(x,y) \in \mathbb{R}^2$. Consider $\gamma(t) := (t, 0)$ for $t \in [-1, 1]$, and $\widetilde{\gamma}(t) := (-\cos t, \sin t)$ for $t \in [0, \pi]$. The initial point of both γ and $\widetilde{\gamma}$ is $(-1, 0)$

and the final point of both γ and $\widetilde{\gamma}$ is $(1,0)$. However,

$$\int_{\gamma} \boldsymbol{F}\cdot d\boldsymbol{s} \;=\; \int_{-1}^{1}(t-0,t)\cdot(1,0)dt = \int_{-1}^{1} t\,dt = 0,$$

whereas

$$\begin{aligned}\int_{\widetilde{\gamma}} \boldsymbol{F}\cdot d\boldsymbol{s} &= \int_{0}^{\pi}(-\cos t-\sin t,-\cos t)\cdot(\sin t,\cos t)dt\\ &= \int_{0}^{\pi}(-\cos t\sin t-\sin^2 t-\cos^2 t)dt\\ &= -\frac{1}{2}\int_{0}^{\pi}\sin 2t dt-\int_{0}^{\pi}dt = 0-\pi=-\pi.\end{aligned}$$

Hence line integrals of $\boldsymbol{F}$ are not path-independent in $\mathbb{R}^2$.

We now give a criterion for the path-independence of line integrals of a vector field.

Proposition (Criterion for Path Independence)

Let $\boldsymbol{F}$ be a continuous vector field on $D \subset \mathbb{R}^m$. Then line integrals of $\boldsymbol{F}$ are path-independent in $D \iff$ the line integral of $\boldsymbol{F}$ along every closed path that lies in D is zero.

$\Longrightarrow$) Let $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$ be a closed path that lies in D. Then $\boldsymbol{\gamma}(\alpha) = \boldsymbol{\gamma}(\beta) \in D$. Define $\boldsymbol{\gamma}_0(t) := \boldsymbol{\gamma}(\alpha)$ for all $t \in [\alpha, \beta]$. Then both $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_0$ lie in D, and they have the same initial point $\boldsymbol{\gamma}(\alpha)$ as well as the same final point $\boldsymbol{\gamma}(\alpha)$. By the path-independence of $\boldsymbol{F}$ in D,

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\gamma}_0} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\alpha}^{\beta} \boldsymbol{F}(\boldsymbol{\gamma}(\alpha)) \cdot \boldsymbol{0}\, dt = 0.$$

$\Longleftarrow$) Let $\boldsymbol{\gamma}$ and $\widetilde{\boldsymbol{\gamma}}$ be paths lying in D which have the same initial point, and also the same final point. Then $\boldsymbol{\gamma}_c := \boldsymbol{\gamma} + (-\widetilde{\boldsymbol{\gamma}})$ is a closed path in D, and so

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} - \int_{\widetilde{\boldsymbol{\gamma}}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{-\widetilde{\boldsymbol{\gamma}}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\gamma}_c} \boldsymbol{F} \cdot d\boldsymbol{s} = 0. \quad \square$$

FTC for Line Integrals

Consider the following versions of Part I and Part II of the Fundamental Theorem of Calculus for Riemann integrals of scalar fields on an open interval in $\mathbb{R}$.

Part I: Let I be an open interval in $\mathbb{R}$, and let g be a continuous scalar field on I. If $a \in I$, and we let $f(x) := \int_a^x g(t)dt$ for $x \in I$, then $f' = g$ on I.

Part II: Let I be an open interval in $\mathbb{R}$, and let f be a smooth scalar field on I. Then

$$\int_a^b f'(x)dx = f(b) - f(a) \quad \text{for all } a, b \in I.$$

We shall now attempt to find analogues of the above results for line integrals of vector fields. In this process, we shall obtain another important criterion for the path-independence of line integrals of a vector field.

Theorem (FTC for Line Integrals: Part I)

Let D be an open subset of $\mathbb{R}^m$, and let $\boldsymbol{G}$ be a continuous vector field on D. Suppose line integrals of $\boldsymbol{G}$ are path-independent in D. If D is path-connected and for a fixed $\boldsymbol{x}_0$ in D, we define

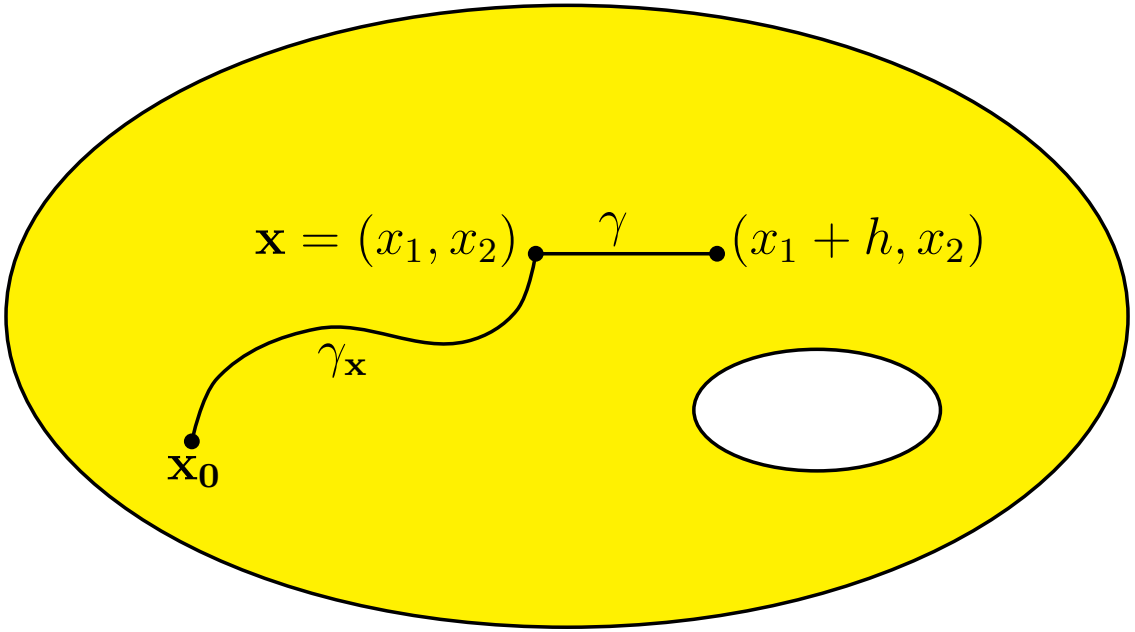
$$f(\boldsymbol{x}) := \int_{\gamma_{\boldsymbol{x}}} \boldsymbol{G} \cdot d\boldsymbol{s}, \quad \text{where } \gamma_{\boldsymbol{x}} \text{ is a path in } D \text{ from } \boldsymbol{x}_0 \text{ to } \boldsymbol{x},$$

then $\nabla f = \boldsymbol{G}$ on D. Consequently, $\boldsymbol{G}$ is a gradient field on D.

Proof: Suppose D is path-connected. Then for every $\boldsymbol{x} \in D$, there is a (piecewise smooth) path $\gamma_{\boldsymbol{x}}$ with initial point $\boldsymbol{x}_0$ and final point $\boldsymbol{x}$. Further, as line integrals of $\boldsymbol{G}$ are path-independent on D, the function $f : D \to \mathbb{R}$ mentioned above is well-defined.

For simplicity, let $m = 2$, and consider $\boldsymbol{x} := (x_1, x_2) \in D$.

Since the set D is open in $\mathbb{R}^2$, the line segment joining (x_1, x_2) to $(x_1 + h, x_2)$ lies in D for all $h \in \mathbb{R}$ if $|h|$ is sufficiently small.



Let $\gamma(t) := (x_1 + th, x_2)$ for $t \in [0, 1]$. Now $\gamma_{\boldsymbol{x}}$ followed by γ is a (piecewise smooth) path from $\boldsymbol{x}_0$ to $(x_1 + h, x_2)$. Hence

$$f(x_1+h, x_2)-f(x_1, x_2) = \int_\gamma \boldsymbol{G}\cdot d\boldsymbol{s} = \int_0^1 \boldsymbol{G}(x_1+th, x_2)\cdot(h, 0)dt.$$

Let $\boldsymbol{G} := (G_1, G_2)$. It follows that for all $h \neq 0$ and small $|h|$,

$$\frac{f(x_1 + h, x_2) - f(x_1, x_2)}{h} = \int_0^1 G_1(x_1 + th, x_2)dt.$$

Also, since G_1 is continuous at (x_1, x_2),

$$\int_0^1 G_1(x_1+th, x_2)dt \to \int_0^1 G_1(x_1, x_2)dt = G_1(x_1, x_2) \text{ as } h \to 0.$$

Thus the partial of f at any $(x_1, x_2) \in D$ with respect to the first variable is equal to $G_1(x_1, x_2)$. Similarly, the partial of f at (x_1, x_2) with respect to the second variable equals $G_2(x_1, x_2)$. Hence $\nabla f = (f_{x_1}, f_{x_2}) = (G_1, G_2) = \boldsymbol{G}$.

Now if D is any open subset of $\mathbb{R}^2$, then by considering components of D, each of which is open and path-connected, we see that $\nabla f = \boldsymbol{G}$. □

Theorem (FTC for Line Integrals: Part II)

Let D be an open subset of $\mathbb{R}^m$, and let f be a smooth scalar field on D. Let γ be a path that lies in D, A be the initial point of γ, and B be the final point of γ. Then

$$\int_{\gamma} (\nabla f) \cdot d\boldsymbol{s} = f(B) - f(A).$$

Consequently, line integrals of a continuous gradient field on D are path-independent in D.

Proof: By part (ii) of the chain rule,

$$\int_{\gamma} (\nabla f) \cdot d\boldsymbol{s} = \int_{\alpha}^{\beta} (\nabla f)(\gamma(t)) \cdot \gamma'(t) dt = \int_{\alpha}^{\beta} (f \circ \gamma)'(t) dt.$$

So by Part II of the FTC for the function $f \circ \gamma : [\alpha, \beta] \to \mathbb{R}$,

$$\int_{\gamma} \nabla f \cdot d\boldsymbol{s} = f(\gamma(\beta)) - f(\gamma(\alpha)) = f(B) - f(A). \quad \square$$

The previous result shows that if $\boldsymbol{G}$ is a continuous gradient field on an open subset of $\mathbb{R}^m$, then the evaluation of a line integral $\int_\gamma \boldsymbol{G}\cdot d\boldsymbol{s}$ can be simplified on replacing γ by a suitable path having the same initial point and the same final point.

If we actually know a scalar field f such that $\nabla f = \boldsymbol{G}$, then the evaluation of $\int_\gamma \boldsymbol{G}\cdot d\boldsymbol{s}$ is immediate even when the path γ is very complicated. This process is similar to the evaluation of a Riemann integral by finding an antiderivative of its integrand.

Example: Let $\boldsymbol{G}(x,y,z) := (yz, zx, xy)$ for $(x,y,z) \in \mathbb{R}^3$, and let $\gamma(t) := (\cos^4 t, \sin^4 t, \tan^4 t)$, $t \in [0, \pi/4]$. If we let $f(x,y,z) := xyz$ for $(x,y,z) \in \mathbb{R}^3$, then $\nabla f = \boldsymbol{G}$ on $\mathbb{R}^3$, and

$$\int_\gamma \boldsymbol{G}\cdot d\boldsymbol{s} = f(\gamma(\pi/4)) - f(\gamma(0)) = f\left(\frac{1}{4}, \frac{1}{4}, 1\right) - f(1,0,0) = \frac{1}{16}.$$

Parts I and II of the FTC for line integrals show that line integrals of a continuous vector field $\boldsymbol{G}$ on an open subset of $\mathbb{R}^m$ are path-independent if and only if $\boldsymbol{G}$ is a gradient field.

In view of the results proved above, it is important to know whether a continuous vector field $\boldsymbol{F}$ is indeed a gradient field. First we prove a necessary condition when $\boldsymbol{F}$ is smooth.

Proposition (Cross-Derivative Test)

Let $D \subset \mathbb{R}^m$ be open, and let $\boldsymbol{F} := (F_1, \ldots, F_m) : D \to \mathbb{R}^m$ be a smooth vector field. Suppose $\boldsymbol{F}$ is a gradient field. Then $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}$ for all $i, j = 1, \ldots, m$.

Proof: Let $\boldsymbol{F} = \nabla f$ on D, where $f : D \to \mathbb{R}$ is smooth. Since the vector field $\boldsymbol{F}$ is also smooth, the partial derivatives of f of orders 1 and 2 are continuous on D. By the Mixed Partials Theorem,

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial F_j}{\partial x_i} \quad \text{for } i, j = 1, \ldots, m. \qquad \square$$

The above test says that 'cross partials' are equal.

Special Cases of the Cross-Derivative Test

$m := 2$, $\boldsymbol{F} := (P, Q)$: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

$m := 3$, $\boldsymbol{F} := (P, Q, R)$: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}, \dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}, \dfrac{\partial R}{\partial x} = \dfrac{\partial P}{\partial z}$,

that is, $\operatorname{curl} \boldsymbol{F} = \operatorname{curl}(\operatorname{grad} f) = \boldsymbol{0}$.

Examples:

(i) Let $D := \mathbb{R}^2$, and let $\boldsymbol{F} := (P, Q)$, where $P(x, y) := -y$ and $Q(x, y) := x$ for $(x, y) \in D$. Then $P_y = -1$ and $Q_x = 1$. By the cross-derivative test, $\boldsymbol{F}$ is not a gradient field on D.

(ii) Even if $\boldsymbol{F}$ satisfies the cross-derivative test on an open subset D of $\mathbb{R}^m$, $\boldsymbol{F}$ may not be a gradient field. For instance, let $D := \{(x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0)\}$, and $\boldsymbol{F} := (P, Q)$, where $P := -y/(x^2 + y^2)$ and $Q := x/(x^2 + y^2)$, $(x, y) \in D$. Then $P_y = (y^2 - x^2)/(x^2 + y^2)^2 = Q_x$. But $\boldsymbol{F}$ is not a gradient field on D: If γ is the standard unit circle in D, then

$$\int_\gamma \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{-\pi}^{\pi} (-\sin t, \cos t)\cdot(-\sin t, \cos t)dt = 2\pi \neq 0.$$

Next, let $D := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \neq (0, 0)\}$, and let $\boldsymbol{F} := (P, Q, R)$, where $P := -y/(x^2 + y^2)$, $Q := x/(x^2 + y^2)$ and $R := 1$ for $(x, y, z) \in D$. As above, $P_y = Q_x$, and so $\text{curl}\,(\boldsymbol{F}) = (R_y - Q_z, P_z - R_x, Q_x - P_y) = \boldsymbol{0}$. But $\boldsymbol{F}$ is not a gradient field on D: If $\gamma(t) := (\cos t, \sin t, 0)$, $t \in [-\pi, \pi]$, then

$$\int_\gamma \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{-\pi}^{\pi} (-\sin t, \cos t, 1)\cdot(-\sin t, \cos t, 0)dt = 2\pi \neq 0.$$

Remark

If $D := \mathbb{R}^m$ and if $\boldsymbol{F}$ satisfies the cross-derivative test on $\mathbb{R}^m$, then it can be proved that $\boldsymbol{F}$ is indeed a gradient field on D. In fact, this also holds if D is an open simply connected subset of $\mathbb{R}^m$ and $\boldsymbol{F}$ satisfies the cross-derivative test on D. Roughly speaking, D is simply connected if it does not have any holes!

Let $D \subset \mathbb{R}^2$. Then D is called **simply connected** if every simple closed path lying in D encloses points of D only. If D is bounded, then it is simply connected if and only if the complement $\mathbb{R}^2 \setminus D$ of D in $\mathbb{R}^2$ is path-connected.
For example, a disk or a rectangle in $\mathbb{R}^2$ is simply connected, but an annulus $\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$ is not.

Let $D \subset \mathbb{R}^3$. Then D is called **simply connected** if every simple closed path lying in D is the 'edge' of a bounded surface lying entirely in D. For example, a ball or a cuboid in $\mathbb{R}^3$ is simply connected, and so is the solid body $D := \{(x, y, z) \in \mathbb{R}^3 : 1 < x^2 + y^2 + z^2 < 4\}$, but the set $\{(x, y, z) \in \mathbb{R}^3 : (x, y) \neq (0, 0)\}$ is not.

Let a smooth vector field $\boldsymbol{F}$ satisfy the cross-derivative test on an open simply connected subset D of $\mathbb{R}^2$ or $\mathbb{R}^3$. The theorems of Green and Stokes will show that $\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = 0$ for every simple closed path lying in D, and hence for every closed path lying in D. This implies that $\boldsymbol{F}$ is a gradient field on D.

As a special case, let $D := I \times J \times K$, where I, J, K are open intervals in $\mathbb{R}$. Suppose a smooth vector field $\boldsymbol{F} := (P, Q, R)$ on D satisfies the cross-derivative test, that is, $\operatorname{curl}(\boldsymbol{F}) = \boldsymbol{0}$ on D. Then $\boldsymbol{F}$ is a gradient field. How to find a scalar field f on D such that $\boldsymbol{F} = \nabla f$, that is, $(P, Q, R) = (f_x, f_y, f_z)$? This can be accomplished by repeated integration as follows. Fix $x_0 \in I$.

Solve $f_x = P$ by integrating with respect to x: Let

$$f(x, y, z) := \int_{x_0}^{x} P(u, y, z)\,du + g(y, z) \quad \text{for } (x, y, z) \in D,$$

where g is an arbitrary function of $(y, z) \in J \times K$. Use this expression for f in the equation $f_y = Q$ to obtain an equation for g_y. Solve it by integrating with respect to y. Its solution will involve an arbitrary function h of $z \in K$. Use this expression for g in the equation $f_z = R$ to obtain an equation for h'. Solve it by integrating with respect to z. Its solution will involve an arbitrary constant. This yields f with $\nabla f = \boldsymbol{F}$.

Further, if $\boldsymbol{F} = \nabla \tilde{f}$ for a scalar field $\tilde{f}$ on D as well, then $\nabla(\tilde{f} - f) = 0$ on D. By the Trivariate Mean Value Theorem, $\tilde{f} - f = c$, that is, $\tilde{f} = f + c$, where c is a constant.

Example:
Let $\boldsymbol{F} := (P, Q, R)$ where $P := x^2 + y\,z$, $Q := y^2 + z\,x$, $R := z^2 + x\,y$ for $(x, y, z) \in \mathbb{R}^3$. It is easy to check that $\operatorname{curl}(\boldsymbol{F}) = \boldsymbol{0}$. Let f be a scalar field such that $\nabla f = \boldsymbol{F}$. Then

$f_x = P = x^2 + y\,z \implies f(x, y, z) = x^3/3 + x\,y\,z + g(y, z)$.
$f_y = Q = y^2 + z\,x \implies x\,z + g_y = y^2 + z\,x$, and so $g_y = y^2$.
Thus $g(y, z) = y^3/3 + h(z)$, and as a consequence,

$$f(x, y, z) = \frac{x^3 + y^3}{3} + x\,y\,z + h(z).$$

$f_z = R = z^2 + x\,y \implies x\,y + h' = z^2 + x\,y$, and so $h' = z^2$.
Thus $h(z) = z^3/3 + c$, and as a consequence,

$$f(x, y, z) = \frac{x^3 + y^3 + z^3}{3} + x\,y\,z + c, \quad \text{where } c \text{ is a constant.}$$

Let γ be a simple closed piecewise smooth path in $\mathbb{R}^2$, and let $C := \gamma([\alpha, \beta])$ denote the geometric curve in $\mathbb{R}^2$ defined by γ. Suppose $\boldsymbol{F}$ is a continuous vector field defined on C.

Since the line integral of a continuous vector field along a smooth path is invariant under a reparametrization only up to its sign, $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$ is determined only up to its sign. To determine it completely, we specify the sense or the direction in which the geometric curve C is traversed. For example, we may specify that as we travel along the curve C In this case, the curve C is traversed in **anticlockwise** direction (as viewed from high above the curve).

Suppose D is a bounded region in $\mathbb{R}^2$ whose boundary ∂D consists of a finite number of nonintersecting simple closed piecewise smooth geometric curves $C_1, \ldots, C_k$.