MA 105 : Calculus
Division 3, Lecture 34

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Recap of the previous lecture

- Positive orientation for the boundary of a planar region
- Green's theorem; Sketch of proof in a special case
- Computational uses of Green's theorem
- Consequences of Green's theorem:
  - Invariance of some line integrals
  - Alternative formulations of Green's formula

Consequences of the Green Theorem

Proposition

Let $\boldsymbol{F} := (P, Q)$ be a smooth vector field on an open subset containing a closed and bounded subset D of $\mathbb{R}^2$ such that $Q_x = P_y$ on D. If ∂D consists of a finite number of nonintersecting simple closed piecewise smooth curves, and if ∂D is positively oriented, then

$$\int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\partial D} P\ dx + Q\ dy = 0.$$

Proof

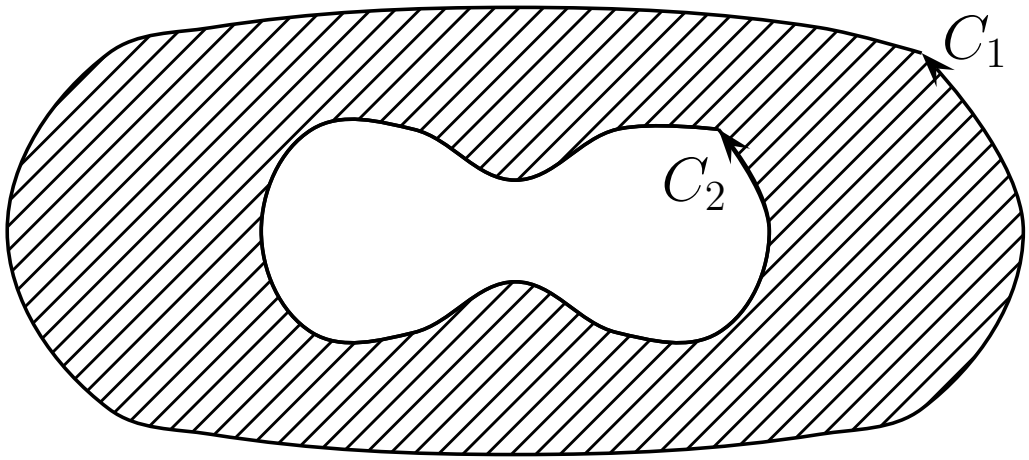
By the Green theorem,

$$\int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_D (Q_x - P_y) d(x, y) = \iint_D 0\, d(x, y) = 0. \quad \square$$

Invariance of Some Line Integrals

Suppose C_1 and C_2 are simple closed piecewise smooth curves such that C_2 lies in the interior of C_1, that is, C_1 encloses C_2, and suppose both are oriented counterclockwise. Let P, Q be smooth scalar fields satisfying $Q_x = P_y$ on an open set containing C_1, C_2 and the region between them. Then

$$\int_{C_1} P\ dx + Q\ dy = \int_{C_2} P\ dx + Q\ dy.$$



This result follows by applying the Green theorem to the subset D of $\mathbb{R}^2$ consisting of C_1, C_2 and the region between them, and by noting that $\partial D = C_1 + (-C_2)$.

This is known as the deformation principle for line integrals along paths in $\mathbb{R}^2$.

Example: (Gauss Law in $\mathbb{R}^2$): Let C be a piecewise smooth simple closed geometric curve in $\mathbb{R}^2$, and let D denote the union of C and the region enclosed by C, so that $\partial D = C$. Suppose $(0,0) \notin C$, and C is positively oriented. Then

$$\int_C \frac{\tilde{\boldsymbol{r}}}{r^2} \cdot d\boldsymbol{s} = \begin{cases} 0 & \text{if } (0,0) \notin D, \\ 2\pi & \text{if } (0,0) \in D, \end{cases}$$

where $\tilde{\boldsymbol{r}}(x,y) := (-y,x)$ and $r(x,y) := \|(x,y)\|$, $(x,y) \in \mathbb{R}^2$.

Proof. Let $\boldsymbol{F} := \tilde{\boldsymbol{r}}/r^2$, that is, $\boldsymbol{F} := (P,Q)$, where $P = -y/r^2 = -y/(x^2+y^2)$ and $Q := x/r^2 = x/(x^2+y^2)$ for $\boldsymbol{r} \neq \boldsymbol{0}$. Then $Q_x - P_y = 0$ on $D \setminus \{(0,0)\}$ as we saw earlier.

First, let $(0,0) \notin D$. Then the vector field $\boldsymbol{F}$ is smooth on D. Hence $\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_D (Q_x - P_y) d(x,y) = 0$ by the Green theorem.

Next, let $(0,0) \in D$. We have seen earlier that $\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} = 2\pi$, where C_1 denotes the standard parametrized unit circle in $\mathbb{R}^2$ (which is positively oriented). Since $(0,0)$ is an interior point of D, there is $\epsilon > 0$ such that the closed disk of radius ϵ and center $(0,0)$ lies inside D. Let C_ϵ denote the standard parametrized circle of radius ϵ (which is also positively oriented). Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{C_\epsilon} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} = 2\pi$$

by the deformation principle for line integrals, applied first to the curves C and C_ϵ, and then to the curves C_ϵ and C_1. □

Proposition

Let Ω be a simply connected open subset of $\mathbb{R}^2$, and let $\boldsymbol{F} := (P, Q)$ be a smooth vector field such that $Q_x = P_y$ on Ω. Then $\boldsymbol{F}$ is a gradient field on Ω.

Proof. Let C denote a simple closed smooth curve in Ω. Since Ω is simply connected, C encloses points of Ω only. Let D denote the set of points on and inside C. By the Green theorem, $\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_D (Q_x - P_y) d(x, y) = 0$. This also holds if a smooth closed curve C in Ω intersects itself, since we can break up C into several simple closed smooth curves. As a result, line integrals of $\boldsymbol{F}$ are path-independent in D. □

Recall the differential notation for a regular smooth curve γ:

$ds = \|\gamma'(t)\| dt$, $d\boldsymbol{s} = \gamma'(t) dt$, $\hat{\boldsymbol{t}} = \gamma'(t)/\|\gamma'(t)\|$, $d\boldsymbol{s} = \hat{\boldsymbol{t}}\, ds$,

$\hat{\boldsymbol{n}} = \hat{\boldsymbol{t}} \times \boldsymbol{k}$. If $\boldsymbol{F} := (P, Q)$, then $\boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, ds = P\, dy - Q\, dx$.

Alternative Formulations of the Green Formula

We can express the conclusion of the Green theorem, namely,

$$\int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d(x,y).$$

as follows.

(i) Circulation-Curl Form or Tangential Form:

$$\int_{\partial D} \boldsymbol{F} \cdot \hat{\boldsymbol{t}}\, ds = \iint_D (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{k}\, d(x,y). \quad [\text{Let } \boldsymbol{F} := (P, Q, 0).]$$

(The Stokes theorem is a 3-dimensional curved version of this.)

(ii) Flux-Divergence Form or Normal Form:

$$\int_{\partial D} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, ds = \iint_D (\operatorname{div} \boldsymbol{F}) d(x,y). \quad [\text{Let } \boldsymbol{F} := (-Q, P, 0).]$$

(The Gauss theorem is a 3-dimensional analogue of this.)

Surfaces in $\mathbb{R}^3$

We now move up one dimension, and pass from paths and line integrals to surfaces and surface integrals.

Just as a curve in $\mathbb{R}^2$ is given implicitly by an equation $F(x, y) = 0$, or parametrically by $(x(t), y(t))$ for $t \in [\alpha, \beta]$, a surface in $\mathbb{R}^3$ is given implicitly by an equation $F(x, y, z) = 0$, or parametrically by $(x(u, v), y(u, v), z(u, v))$ for $(u, v) \in E$, where E is a subset of $\mathbb{R}^2$. Here u and v are independent parameters, that is, they do not depend on each other.

For example, the **unit sphere** in $\mathbb{R}^3$ is given implicitly by the equation $x^2 + y^2 + z^2 - 1 = 0$, and also parametrically by $\big((\sin u)(\cos v), (\sin u)(\sin v), \cos u\big)$ for $(u, v) \in [0, \pi] \times [-\pi, \pi]$.

We have discussed the tangent planes and normal lines for implicitly defined surfaces in Lecture 24. Now we shall treat parametrically defined surfaces.

Parametrized Surfaces

A **parametrized surface** in $\mathbb{R}^3$ is a continuous map $\Phi : E \to \mathbb{R}^3$, where E is a path-connected subset of $\mathbb{R}^2$ having an area. Let $\Phi(u, v) := (x(u, v), y(u, v), z(u, v))$ for $(u, v) \in E$. The parametrized surface Φ is called **smooth** if the functions x, y, z have continuous partial derivatives on an open subset of $\mathbb{R}^2$ containing E.

Given a parametrized surface, we may eliminate the parameters u, v to obtain an equation involving only x, y, z, which defines the surface implicitly. The converse process is generally harder.

Examples

(i) Consider a continuous function $f : E \to \mathbb{R}$, and define $\Phi : E \to \mathbb{R}^3$ by $\Phi(x, y) := (x, y, f(x, y))$ for $(x, y) \in E$. This parametrized surface is called the **graph** of f. It is implicitly given by the equation $F(x, y, z) := z - f(x, y) = 0$.

(ii) Let $a > 0$ and $E := [-\pi, \pi] \times [0, h]$, where $h > 0$.

Define $\Phi : E \to \mathbb{R}^3$ by $\Phi(\theta, z) := (a\cos\theta, a\sin\theta, z)$ for $(\theta, z) \in E$. This parametrized surface is called the right circular **cylinder** of radius a and height h. It is implicitly given by the equation $F(x, y, z) := x^2 + y^2 - a^2 = 0$ with $0 \leq z \leq h$.

Define $\Phi : E \to \mathbb{R}^3$ by $\Phi(\theta, z) := (z\cos\theta, z\sin\theta, z)$ for $(\theta, z) \in E$. This parametrized surface is called the right circular **cone** of height h. It is implicitly given by the equation $F(x, y, z) := x^2 + y^2 - z^2 = 0$ with $0 \leq z \leq h$.

(iii) Let $a > 0$ and $E := [0, \pi] \times [-\pi, \pi]$. Define $\Phi : E \to \mathbb{R}^3$ by $\Phi(\varphi, \theta) := (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$ for $(\varphi, \theta) \in E$. This parametrized surface is called the **standard sphere of radius** a. It is implicitly given by the equation $F(x, y, z) := x^2 + y^2 + z^2 - a^2 = 0$.

Normal Vector to a Parametrized Surface

Let $\Phi : E \to \mathbb{R}^3$ be a smooth parametrized surface. Let $(u_0, v_0) \in E$, and let $P_0 := \Phi(u_0, v_0)$.

Suppose $\gamma(t) := (u(t), v(t))$, $t \in [\alpha, \beta]$, is a smooth path that lies in E and passes through (u_0, v_0), and let $t_0 \in [\alpha, \beta]$ be such that $\gamma(t_0) = (u_0, v_0)$.

Now the path $\Phi \circ \gamma$ lies in $\Phi(E)$, and by the Chain Rule, the tangent vector to this path at P_0 is

$$(\Phi \circ \gamma)'(t_0) = \Phi_u(u_0, v_0)\, u'(t_0) + \Phi_v(u_0, v_0)\, v'(t_0).$$

Because $\Phi_u(u_0, v_0) \times \Phi_v(u_0, v_0)$ is perpendicular to both $\Phi_u(u_0, v_0)$ and $\Phi_v(u_0, v_0)$, it is also perpendicular to the tangent vector $(\Phi \circ \gamma)'(t_0)$. Since this holds for any such path γ that lies in E, $(\Phi_u \times \Phi_v)(u_0, v_0) := \Phi_u(u_0, v_0) \times \Phi_v(u_0, v_0)$ is called a **normal vector** to Φ at P_0. It is also called the **fundamental product** for Φ at $(u_0, v_0) \in E$.

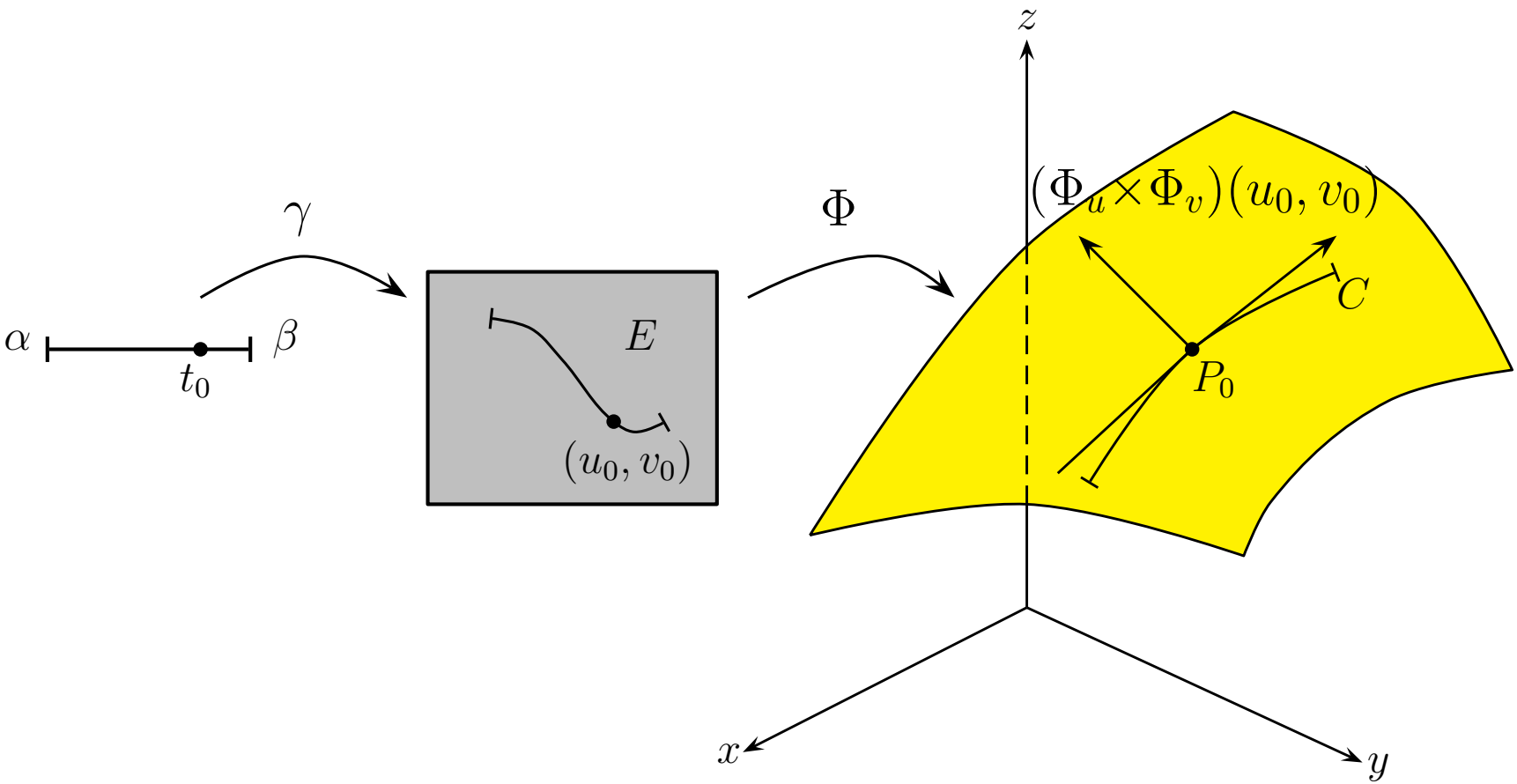


Figure: Normal vector to a parametrized surface at P_0

The Fundamental Product

How do we find the fundamental product of a parametrized surface $\Phi : E \to \mathbb{R}^3$ at $(u, v) \in E$?
Let $\Phi(u, v) := (x(u, v), y(u, v), z(u, v))$ for $(u, v) \in E$. Then

$$\begin{aligned}
\Phi_u \times \Phi_v &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \\
&= \left(\begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}, \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix}, \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \right) \\
&= \left(\frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}, \frac{\partial(x, y)}{\partial(u, v)} \right),
\end{aligned}$$

where $\dfrac{\partial(y, z)}{\partial(u, v)}$ is the Jacobian of the transformation $(u, v) \longmapsto (y(u, v), z(u, v))$ etc., as defined in Lecture 28.

We observe that

$$\|(\Phi_u \times \Phi_v)(u,v)\| = \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}.$$

A smooth parametrized surface Φ is said to be **regular** if $(\Phi_u \times \Phi_v)(u,v) \neq \mathbf{0}$ for every $(u,v) \in E$.
In this case,

$$\hat{\boldsymbol{n}}\,(u,v) := \frac{(\Phi_u \times \Phi_v)(u,v)}{\|(\Phi_u \times \Phi_v)(u,v)\|}$$

is called a **unit normal vector** to Φ at $\Phi(u,v)$, $(u,v) \in E$.

Let $(u_0,v_0) \in E$, and let $P_0 := \Phi(u_0,v_0)$. If $P_0 := (x_0,y_0,z_0)$, then the **tangent plane** to Φ at P_0 is given by the equation

$$(\Phi_u \times \Phi_v)(u_0,v_0) \cdot (x-x_0, y-y_0, z-z_0) = 0,$$

that is, by the equation $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$,

where $(a,b,c) := (\Phi_u \times \Phi_v)\,(u_0,v_0)$.

Examples

(i) Graph of a function

Let E be a path-connected subset of $\mathbb{R}^2$ having an area, and let $f : E \to \mathbb{R}$ be a smooth function.
Let $\Phi(x, y) = (x, y, f(x, y))$ for $(x, y) \in E$. Then

$$\Phi_x \times \Phi_y = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x, -f_y, 1) \quad \text{on } E.$$

Since $\|\Phi_x \times \Phi_y\| = \sqrt{f_x^2 + f_y^2 + 1} \neq 0$ on E, Φ is a regular surface. The tangent plane to Φ at $(x_0,y_0,f(x_0, y_0))$ is given by

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

(ii) Cylinder

Let $\Phi(\theta, z) := (a\cos\theta, a\sin\theta, z)$ for $(\theta, z) \in [-\pi, \pi] \times [0, h]$, where $a > 0$. Then

$$\Phi_\theta \times \Phi_z = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -a\sin\theta & a\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (a\cos\theta, a\sin\theta, 0).$$

Since $\|\Phi_\theta \times \Phi_z\| = a \neq 0$ on $(\theta, z) \in [-\pi, \pi] \times [0, h]$, Φ is a regular surface. The tangent plane to Φ at $(a\cos\theta_0, a\sin\theta_0, z_0)$ is given by

$$(\cos\theta_0)x + (\sin\theta_0)y = a.$$

(iii) Sphere

Let $\Phi(\varphi, \theta) := (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$ for $(\varphi, \theta) \in [0, \pi] \times [-\pi, \pi]$, where $a > 0$. Then

$$\Phi_\varphi \times \Phi_\theta = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}.$$

Thus

$$\begin{aligned}\Phi_\varphi \times \Phi_\theta &= (a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\sin\varphi\cos\varphi)\\ &= (a\sin\varphi)\,\Phi(\varphi,\theta).\end{aligned}$$

Since $\|\Phi(\varphi,\theta)\| = a$ for $(\varphi,\theta) \in [0,\pi]\times[-\pi,\pi]$, we obtain

$$\|\Phi_\varphi \times \Phi_\theta\| = a^2\sin\varphi \neq 0 \quad \text{if } (\varphi,\theta) \in (0,\pi)\times[-\pi,\pi],$$

and then the tangent plane to Φ at $\Phi(\varphi_0,\theta_0)$ is given by

$$(\sin\varphi_0\cos\theta_0)x + (\sin\varphi_0\sin\theta_0)y + (\cos\varphi_0)z = a.$$

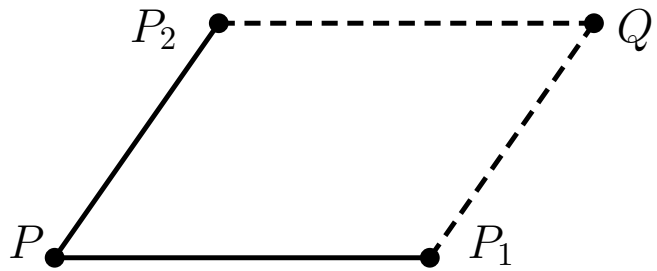
Using the parametrization $\Phi(x,y) := \left(x, y, \sqrt{a^2-x^2-y^2}\right)$ for $(x,y) \in \mathbb{R}^2$ satisfying $x^2+y^2 \leq a^2$, we obtain the unit normal vector at the **north pole** $P_0 := (0,0,a)$ to be $(0,0,1)$, and the equation of the tangent plane at P_0 to be $z = a$. Similarly we may treat the **south pole** $Q_0 := (0,0,-a)$.

Surface Area

Let $\Phi : E \to \mathbb{R}^3$ be a smooth parametrized surface, where E is a path-connected subset of $\mathbb{R}^2$ having an area. Let $(u, v) \in E$. For $h, k \in \mathbb{R}$ with $|h|, |k|$ small, consider

$$P := \Phi(u, v), \quad P_1 := \Phi(u + h, v) \approx \Phi(u, v) + h\, \Phi_u(u, v),$$

$$P_2 := \Phi(u, v+k) \approx \Phi(u, v) + k\, \Phi_v(u, v), \quad Q := \Phi(u+h, v+k).$$



Area of the parallelogram with sides PP_1 and PP_2

$$= \|(P_1 - P) \times (P_2 - P)\| \approx \|\Phi_u(u, v) \times \Phi_v(u, v)\|\, |h||k|.$$

In view of this approximation, we define

$$\text{Area}\,(\Phi) := \iint_E \|(\Phi_u \times \Phi_v)(u, v)\| d(u, v).$$

Since the subset E of $\mathbb{R}^2$ has an area, the two-dimensional content of ∂E is zero. Also, the function $\|\Phi_u \times \Phi_v\|$ is continuous on E. Hence the integral in the definition of Area (Φ) is well-defined. In analogy with the differential notation $ds = \|\gamma'(t)\|dt$, we introduce the following differential notation:

$$dS = \|\Phi_u \times \Phi_v\| d(u, v).$$

Examples

(i) Graph of a function

Let a subset E of $\mathbb{R}^2$ have an area, $f : E \to \mathbb{R}$ be a smooth function, and $\Phi(x, y) = (x, y, f(x, y))$ for $(x, y) \in E$. Then

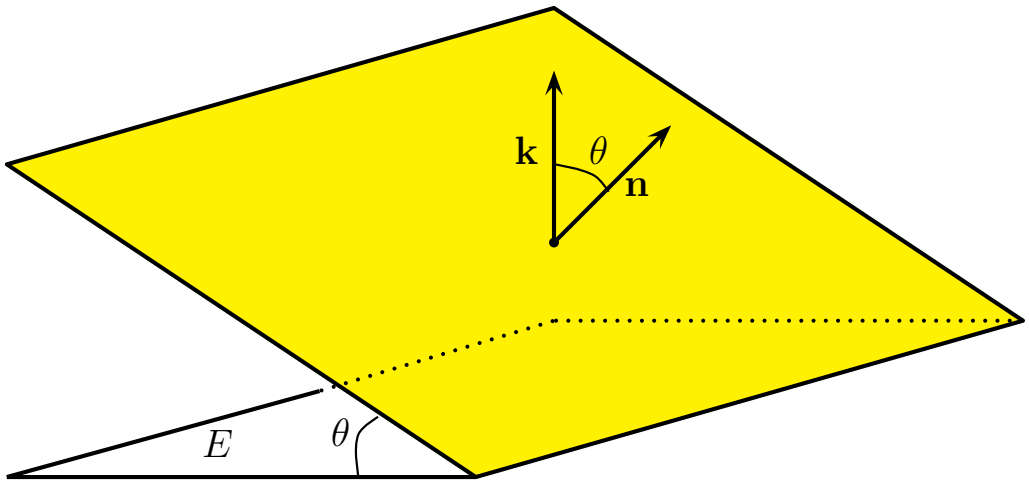
$$\begin{aligned} \text{Area}\,(\Phi) &= \iint_E \|(-f_x, -f_y, 1)\| d(x, y) \\ &= \iint_E \sqrt{1 + f_x^2 + f_y^2}\; d(x, y). \end{aligned}$$

Let $\theta(x,y) \in [0,\pi]$ be the angle between $\Phi_x \times \Phi_y = (-f_x, -f_y, 1)$ at (x,y) and $\boldsymbol{k} = (0,0,1)$. Since $(\Phi_x \times \Phi_y) \cdot \boldsymbol{k} = 1 \neq 0$, we see that $\theta(x,y) \neq \pi/2$ and

$$\cos\theta(x,y) = \frac{1}{\sqrt{1+f_x^2+f_y^2}}, \qquad \text{Area}\,(\Phi) = \iint_E \frac{d(x,y)}{\cos\theta(x,y)}.$$

In case $\theta(x,y) := \theta$, a constant for all $(x,y) \in E$, then $\theta \neq \pi/2$ and we obtain

$$\text{Area}\,(\Phi) = \frac{\text{Area}\,(E)}{\cos\theta} \quad \text{(Area Cosine Principle)}.$$



(ii) Let $E := [-\pi, \pi] \times [0, h]$, $\Phi(\theta, z) := (a\cos\theta, a\sin\theta, z)$, and $\Psi(\theta, z) := (a\cos 2\theta, a\sin 2\theta, z)$ for $(\theta, z) \in E$. Then

$$\begin{aligned}
\text{Area}\,(\Phi) &= \iint_E \|\Phi_\theta \times \Phi_z\|\, d(\theta, z) = \iint_E a\, d(\theta, z) = 2\pi a\, h, \\
\text{Area}\,(\Psi) &= \iint_E \|\Psi_\theta \times \Psi_z\|\, d(\theta, z) = \iint_E 2a\, d(\theta, z) = 4\pi a\, h.
\end{aligned}$$

We note that $\Psi(E) = \Phi(E)$, but $\text{Area}\,(\Psi) = 2\,\text{Area}\,(\Phi)$.

(iii) Let $E := [0, \pi] \times [-\pi, \pi]$, and $\Phi(\varphi, \theta) = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$ for $(\varphi, \theta) \in E$. Then

$$\begin{aligned}
\text{Area}\,(\Phi) &= \iint_E \|\Phi_\varphi \times \Phi_\theta\|\, d(\varphi, \theta) = \iint_E a^2 \sin\varphi\, d(\varphi, \theta) \\
&= \int_{-\pi}^{\pi} \left(\int_0^{\pi} a^2 \sin\varphi\, d\varphi \right) d\theta = 4\pi a^2.
\end{aligned}$$

(iv) Let C be a smooth curve in $\mathbb{R}^2$ given by $\gamma(t) := (x(t), y(t))$, $t \in [\alpha, \beta]$. If C lies on or above the x-axis, and C is revolved about the x-axis, then it generates a surface parametrized by

$$\Phi(t, \theta) := (x(t),\, y(t)\cos\theta,\, y(t)\sin\theta) \quad \text{for } (t, \theta) \in E,$$

where $E := [\alpha, \beta] \times [-\pi, \pi]$. For all $(t, \theta) \in E$,

$$\begin{aligned}(\Phi_t \times \Phi_\theta)(t, \theta) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x'(t) & y'(t)\cos\theta & y'(t)\sin\theta \\ 0 & -y(t)\sin\theta & y(t)\cos\theta \end{vmatrix} \\ &= (y(t)y'(t), -x'(t)y(t)\cos\theta, -x'(t)y(t)\sin\theta).\end{aligned}$$

By the Fubini theorem, we obtain

$$\begin{aligned}\text{Area}\,(\Phi) &= \iint_E \sqrt{y(t)^2 y'(t)^2 + x'(t)^2 y(t)^2}\, d(t, \theta) \\ &= 2\pi \int_\alpha^\beta y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt, \text{ as defined in Lecture 18}\end{aligned}$$