MA 105 : Calculus
Division 3, Lecture 35

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Recap of the previous lecture

- Alternative formulations of Green's formula
- Surfaces in 3-space: Implicitly defined and parametrized surface
- Normal vector to a smooth parametrized surface
- Fundamental product and Jacobians
- Unit normal vector to a regular parametrized surface
- Tangent plane to a smooth parametrized surface
- Examples
- Surface area of a parametrized surface
- Area Cosine Principle

Surface Integral of a Scalar Field

Let E be a subset of $\mathbb{R}^2$ having an area, and let $\Phi : E \to \mathbb{R}^3$ be a smooth parametrized surface. Let $S := \Phi(E)$, and $f : S \to \mathbb{R}$ be a bounded scalar field. The **surface integral** of f **across** Φ is defined by

$$\iint_{\Phi} f \, dS := \iint_{E} f(\Phi(u,v)) \|(\Phi_u \times \Phi_v)(u,v)\| d(u,v),$$

provided the double integral on the right side exists.
In particular, if f is continuous, then $\iint_{\Phi} f \, dS$ is well-defined.

Leting $f := 1$ on S, we see that $\iint_{\Phi} dS = \text{Area}\,(\Phi)$.

The algebraic and the order properties of the double integral imply similar properties of the surface integral of a scalar field.

Example

Let $E := [0, \pi/2] \times [-\pi, \pi]$, and for $(\varphi, \theta) \in E$, let $\Phi(\varphi, \theta) := (a \sin\varphi \cos\theta, a \sin\varphi \sin\theta, a \cos\varphi)$. Then

$$\begin{aligned} \text{Area}\,(\Phi) &= \iint_E \|\Phi_\varphi \times \Phi_\theta\| d(\varphi, \theta) = \iint_E a^2 \sin\varphi \, d(\varphi, \theta) \\ &= \int_{-\pi}^{\pi} \left(\int_0^{\pi/2} a^2 \sin\varphi \, d\varphi \right) d\theta = 2\pi a^2. \end{aligned}$$

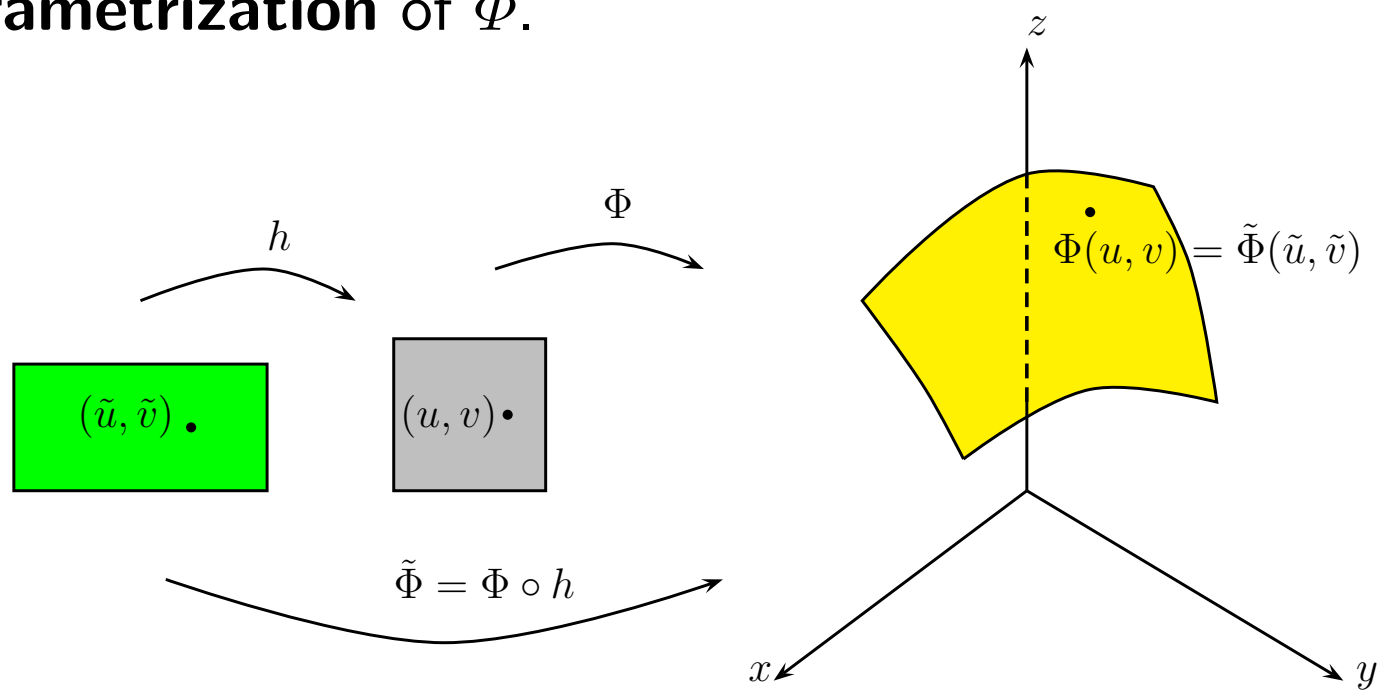
Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) := z$ for $(x, y, z) \in \mathbb{R}^3$. Then

$$\begin{aligned} \iint_\Phi f \, dS &= \int_{-\pi}^{\pi} \left(\int_0^{\pi/2} (a\cos\varphi)(a^2 \sin\varphi) d\varphi \right) d\theta \\ &= \pi a^3 \int_0^{\pi/2} \sin 2\varphi \, d\varphi = \pi a^3. \end{aligned}$$

Reparametrization of a Surface

Let E be a path-connected subset of $\mathbb{R}^2$ having an area, and let $\Phi : E \to \mathbb{R}^3$ be a smooth parametrized surface.

Let $\widetilde{E}$ be a path-connected subset of $\mathbb{R}^2$ having an area, and let $h : \widetilde{E} \to \mathbb{R}^2$ be a continuously differentiable and one-one function such that $h(\widetilde{E}) = E$ and its Jacobian $J(h)$ does not vanish on $\widetilde{E}$. Then the smooth surface $\tilde{\Phi} := \Phi \circ h$ is called a **reparametrization** of Φ.



Proposition

The surface integral of a continuous scalar field over a smooth surface is invariant under reparametrization. In particular, the area of a smooth surface is invariant under reparametrization.

Proof. Let E be a path-connected subset of $\mathbb{R}^2$ having an area. Let $\Phi : E \to \mathbb{R}^3$ be a smooth parametrized surface, and let f be a continuous scalar field on $S := \Phi(E)$.

Let $\tilde{\Phi} := \Phi \circ h$ be a reparametrization of Φ. Using the chain rule, it can be checked that

$$(\widetilde{\Phi}_{\tilde{u}} \times \widetilde{\Phi}_{\tilde{v}})(\tilde{u}, \tilde{v}) = (\Phi_u \times \Phi_v)(h(\tilde{u}, \tilde{v}))J(h)(\tilde{u}, \tilde{v})$$

for all $(\tilde{u}, \tilde{v}) \in \widetilde{E}$.

Then our result on the change of variables formula for double integration gives

$$\begin{aligned}
\iint_{\widetilde{\Phi}} f\,dS &= \iint_{\widetilde{E}} f(\widetilde{\Phi}(\tilde{u},\tilde{v}))\|(\widetilde{\Phi}_{\tilde{u}} \times \widetilde{\Phi}_{\tilde{v}})(\tilde{u},\tilde{v})\|d(\tilde{u},\tilde{v}) \\
&= \iint_{\widetilde{E}} f(\Phi\circ h(\tilde{u},\tilde{v}))\|(\Phi_u \times \Phi_v)(h(\tilde{u},\tilde{v}))\||J(h)(\tilde{u},\tilde{v})|d(\tilde{u},\tilde{v}) \\
&= \iint_{E} f(\Phi(u,v))\|(\Phi_u \times \Phi_v)(u,v)\|d(u,v) \\
&= \iint_{\Phi} f\,dS.
\end{aligned}$$

Considering $f := 1$, we see that the area of a smooth surface is invariant under reparametrization. □

Examples: Let $E := [0, \pi] \times [-\pi, \pi]$, and define
$\Phi(\varphi, \theta) := (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ for $(\varphi, \theta) \in E$.

If $\widetilde{E} := [-\pi, \pi] \times [0, \pi]$, and we define
$\widetilde{\Phi}(\theta, \varphi) := (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ for $(\theta, \varphi) \in \widetilde{E}$,
then $\widetilde{\Phi}$ is a reparametrization of Φ since $\widetilde{\Phi}(\theta, \varphi) = \Phi(h(\theta, \varphi))$,
where $h : \widetilde{E} \to E$ is given by $h(\theta, \varphi) := (\varphi, \theta)$ with $J(h) = -1$.

Similarly, if $\widetilde{E} := [0, \pi/2] \times [-\pi/2, \pi/2]$, and we define
$\widetilde{\Phi}(\varphi, \theta) := (\sin 2\varphi\cos 2\theta, \sin 2\varphi\sin 2\theta, \cos 2\varphi)$ for $(\varphi, \theta) \in \widetilde{E}$,
then $\widetilde{\Phi}$ is a reparametrization of Φ since $\widetilde{\Phi}(\varphi, \theta) = \Phi(h(\varphi, \theta))$,
where $h : \widetilde{E} \to E$ is given by $h(\varphi, \theta) := (2\varphi, 2\theta)$ with $J(h) = 4$.

In both these cases,

$$\iint_{\widetilde{\Phi}} f \, dS = \iint_{\Phi} f \, dS$$

for each continuous scalar field f defined on the unit sphere
$S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.

Geometric Surface

Let E be a subset of $\mathbb{R}^2$ having an area, and let $\Phi : E \to \mathbb{R}^3$ be a parametrized surface. Then $S := \Phi(E)$ is called a **geometric surface**, and Φ is called a **parametrization** of S.

Suppose Φ is smooth and one-one except possibly on a subset of E of two-dimensional content zero. In view of the reparametrization result for continuous scalar fields on parametrized surfaces proved above, we define

$$\iint_S f\, dS := \iint_\Phi f\, dS = \iint_E f(\Phi(u,v))\|(\Phi_u \times \Phi_v)(u,v)\| d(u,v)$$

for a continuous scalar field f on S, and in particular,

$$\text{Area}\,(S) := \text{Area}\,(\Phi) = \iint_E \|(\Phi_u \times \Phi_v)(u,v)\| d(u,v).$$

Surface Integral of a Vector Field

Let E be a subset of $\mathbb{R}^2$ having an area, and $\Phi : E \to \mathbb{R}^3$ be a smooth parametrized surface. Let $\boldsymbol{F}$ be a bounded vector field on $\Phi(E)$. We define the **surface integral** of $\boldsymbol{F}$ **across** Φ by

$$\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} := \iint_{E} \boldsymbol{F}(\Phi(u,v)) \cdot (\Phi_u \times \Phi_v)(u,v)\, d(u,v),$$

provided the double integral on the right exists. In particular, if $\boldsymbol{F}$ is continuous, then $\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S}$ is well-defined.

In analogy with our differential notation $dS := \|\Phi_u \times \Phi_v\| d(u,v)$, we introduce the following differential notation:

$$d\boldsymbol{S} := (\Phi_u \times \Phi_v) d(u,v).$$

Let $\Phi(u,v) := (x(u,v), y(u,v), z(u,v))$ for $(u,v) \in E$. Then

$$\Phi_u \times \Phi_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right).$$

If $\boldsymbol{F} := (P, Q, R)$, then we write the surface integral

$$\iint_\Phi \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_E \left(P \frac{\partial(y,z)}{\partial(u,v)} + Q \frac{\partial(z,x)}{\partial(u,v)} + R \frac{\partial(x,y)}{\partial(u,v)} \right) d(u,v)$$

as $\displaystyle\iint_\Phi P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy$,

with the differential notation $dy \wedge dz := \dfrac{\partial(y,z)}{\partial(u,v)} d(u,v)$, etc.

If the surface Φ is regular, then

$$d\boldsymbol{S} = (\Phi_u \times \Phi_v) d(u,v) = \hat{\boldsymbol{n}} \, \|\Phi_u \times \Phi_v\| d(u,v) = \hat{\boldsymbol{n}} \, dS,$$

and we write the surface integral $\displaystyle\iint_\Phi \boldsymbol{F} \cdot d\boldsymbol{S}$ as $\displaystyle\iint_\Phi \boldsymbol{F} \cdot \hat{\boldsymbol{n}} \, dS$.

The algebraic properties of the double integral imply similar properties of the surface integral of a vector field.

Physical interpretation:

Let $\boldsymbol{F}$ be a vector field on a subset D of $\mathbb{R}^3$. Let $\Phi : E \to \mathbb{R}^3$ be a regular surface such that $\Phi(E) \subset D$. Then $\boldsymbol{F} \cdot \hat{\boldsymbol{n}}$ is the component of $\boldsymbol{F}$ in the direction of the unit normal $\hat{\boldsymbol{n}}$.

If $\boldsymbol{F}$ is an electric field, a magnetic field, or a velocity field, then the surface integral

$$\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{\Phi} \boldsymbol{F} \cdot \hat{\boldsymbol{n}} \, dS$$

represents the flux, that is, the rate of flow (per unit area) of the field $\boldsymbol{F}$ across the surface Φ. As a result, this integral is known as a **flux integral**.

Examples

(i) Let a subset E of $\mathbb{R}^2$ have an area, and let $f : E \to \mathbb{R}$ be a smooth function. Let the smooth parametrized surface $\Phi : E \to \mathbb{R}^3$ represent the graph of f, and let $\boldsymbol{F} : \Phi(E) \to \mathbb{R}^3$ be a continuous vector field. If $\boldsymbol{F} := (P, Q, R)$, then

$$\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{E} (-P\, f_x - Q\, f_y + R) d(x, y)$$

since $d\boldsymbol{S} = (\Phi_x \times \Phi_y) d(x, y) = (-f_x, -f_y, 1) d(x, y)$.

For instance, let $E := [0, 1] \times [0, 1]$, $f(x, y) := x + y + 1$ for $(x, y) \in E$. If $\boldsymbol{F}(x, y, z) := (x^2, y^2, z)$ for $(x, y, z) \in \mathbb{R}^3$, then

$$\begin{aligned}
\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} &= \iint_{E} \big(-x^2 - y^2 + (x + y + 1) \big) d(x, y) \\
&= \int_0^1 \left(\int_0^1 (x + y + 1 - x^2 - y^2) dy \right) dx \\
&= = \frac{1}{2} + \frac{1}{2} + 1 - \frac{1}{3} - \frac{1}{3} = \frac{4}{3}.
\end{aligned}$$

(ii) Let $E := [-\pi, \pi] \times [0, h]$, and $\Phi(\theta, z) := (a\cos\theta, a\sin\theta, z)$ for $(\theta, z) \in E$. If $\boldsymbol{F}(x, y, z) := (y, z, x)$ for $(x, y, z) \in \mathbb{R}^3$, then

$$\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{E} (a^2 \cos\theta \sin\theta + z\, a \sin\theta + 0) d(\theta, z) = 0,$$

since $d\boldsymbol{S} = (\Phi_\theta \times \Phi_z) d(\theta, z) = (a\cos\theta, a\sin\theta, 0) d(\theta, z)$.

(iii) Let $E := [0, \pi] \times [-\pi, \pi]$, and let $\Phi(\varphi, \theta) := (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$ for $(\varphi, \theta) \in E$. If $\boldsymbol{F}(x, y, z) := (x, y, z)$ for $(x, y, z) \in \mathbb{R}^3$, then

$$\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{E} \Phi(\varphi, \theta) \cdot (a\sin\varphi) \Phi(\varphi, \theta) d(\varphi, \theta) = 4a^3\pi,$$

since $d\boldsymbol{S} = (\Phi_\varphi \times \Phi_\theta) d(\varphi, \theta) = a\sin\varphi\, \Phi(\varphi, \theta) d(\varphi, \theta)$.

On the other hand, let $\widetilde{E} := [-\pi, \pi] \times [0, \pi]$, and let $\widetilde{\Phi}(\theta, \varphi) := (a \sin\varphi \cos\theta, a \sin\varphi \sin\theta, a \cos\varphi)$ for $(\theta, \varphi) \in \widetilde{E}$. If $\boldsymbol{F}(x, y, z) := (x, y, z)$ for $(x, y, z) \in \mathbb{R}^3$ as above, then

$$\iint_{\widetilde{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{\widetilde{E}} \widetilde{\Phi}(\theta, \varphi) \cdot (-a \sin\varphi)\, \widetilde{\Phi}(\theta, \varphi) d(\theta, \varphi) = -4a^3\pi,$$

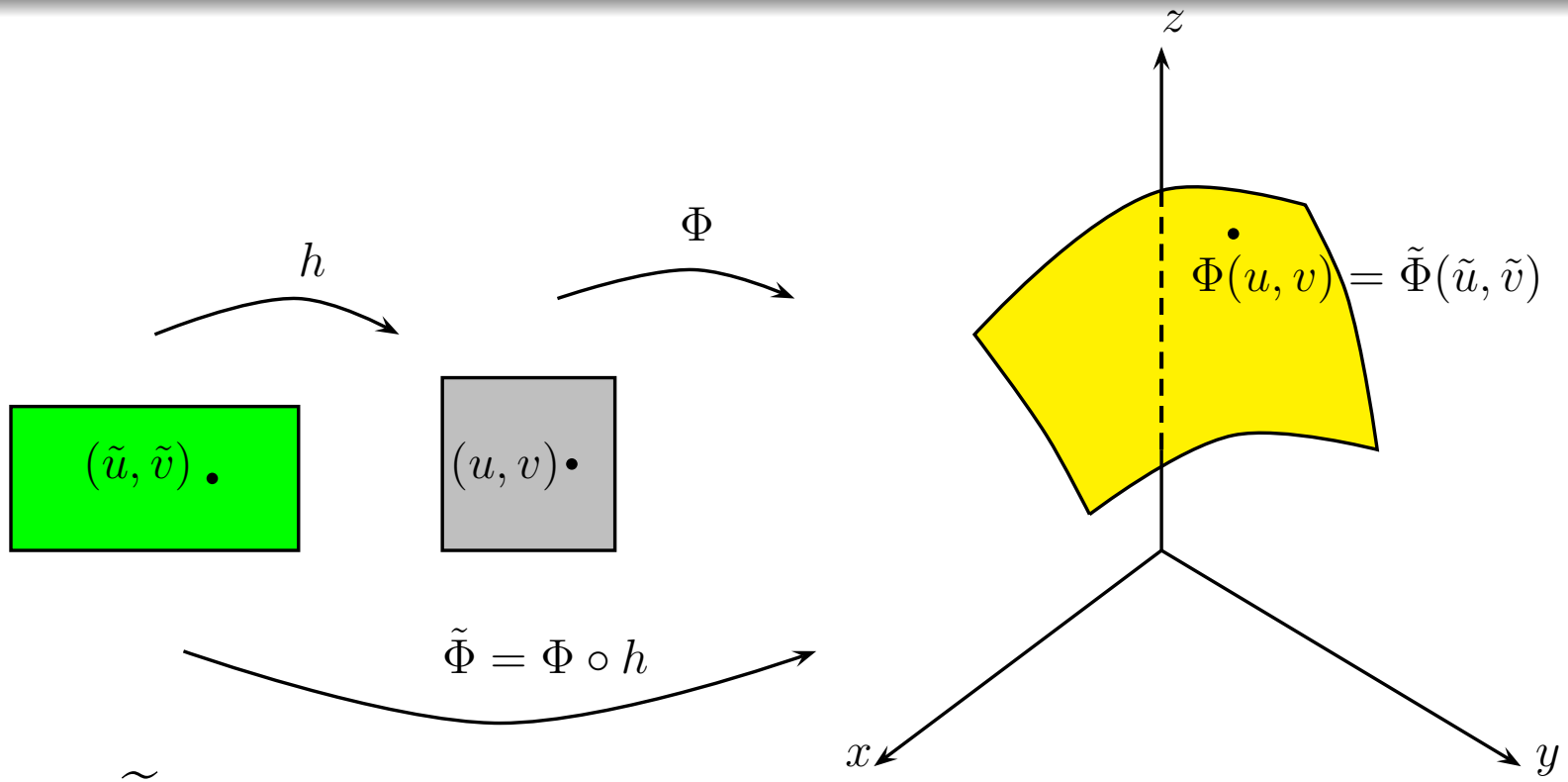
since $d\boldsymbol{S} = (\widetilde{\Phi}_\theta \times \widetilde{\Phi}_\varphi) d(\theta, \varphi) = -a \sin\varphi\, \widetilde{\Phi}(\theta, \varphi) d(\theta, \varphi)$.

Proposition

The surface integral of a continuous vector field across a smooth parametrized surface is invariant under a reparametrization only up to its sign.

Proof. Let $\Phi : E \to \mathbb{R}^3$ be a smooth parametrized surface, and let $\boldsymbol{F}$ be a continuous vector field on $S := \Phi(E)$. Suppose $\widetilde{\Phi} := \Phi \circ h$ be a reparametrization of Φ, where E and $\widetilde{E}$ are path-connected. By the chain rule, for all $(\tilde{u}, \tilde{v}) \in \widetilde{E}$,

$$(\widetilde{\Phi}_{\tilde{u}} \times \widetilde{\Phi}_{\tilde{v}})(\tilde{u}, \tilde{v}) = (\Phi_u \times \Phi_v)(h(\tilde{u}, \tilde{v}))\, J(h)(\tilde{u}, \tilde{v}).$$



Since $\widetilde{E}$ is path-connected, and $J(h)(\tilde{u},\tilde{v}) \neq 0$ for all $(\tilde{u},\tilde{v}) \in \widetilde{E}$, either $J(h)(\tilde{u},\tilde{v}) > 0$ for all $(\tilde{u},\tilde{v}) \in \widetilde{E}$ or $J(h)(\tilde{u},\tilde{v}) < 0$ or all $(\tilde{u},\tilde{v}) \in \widetilde{E}$. By the change of variables formula (which involves $|J(h)(\tilde{u},\tilde{v})|$), we obtain

$\displaystyle\iint_{\widetilde{\Phi}} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{\Phi} \boldsymbol{F}\cdot d\boldsymbol{S}$ in the former case, while

$\displaystyle\iint_{\widetilde{\Phi}} \boldsymbol{F}\cdot d\boldsymbol{S} = -\iint_{\Phi} \boldsymbol{F}\cdot d\boldsymbol{S}$ in the latter. □

Examples

Let $\widetilde{E} := \{(v,u) : (u,v) \in E\}$, and define $h(\tilde{u},\tilde{v}) := (\tilde{v},\tilde{u})$ for $(\tilde{u},\tilde{v}) \in \widetilde{E}$. Then we see that $\widetilde{\Phi}(\tilde{u},\tilde{v}) := \Phi(h(\tilde{u},\tilde{v})) = \Phi(\tilde{v},\tilde{u})$ for $(\tilde{u},\tilde{v}) \in \widetilde{E}$. Since $J(h)(\tilde{u},\tilde{v}) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$ for all $(\tilde{u},\tilde{v}) \in \widetilde{E}$, we obtain

$$\iint_{\widetilde{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} = -\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S}.$$

On the other hand, let $\widetilde{E} := \{(u/2, v/2) : (u,v) \in E\}$, and define $h(\tilde{u},\tilde{v}) := (2\tilde{u}, 2\tilde{v})$ for $(\tilde{u},\tilde{v}) \in \widetilde{E}$. Then we see that $\widetilde{\Phi}(\tilde{u},\tilde{v}) := \Phi(h(\tilde{u},\tilde{v})) = \Phi(2\tilde{u},2\tilde{v})$ for $(\tilde{u},\tilde{v}) \in \widetilde{E}$. Since $J(h)(\tilde{u},\tilde{v}) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$ for all $(\tilde{u},\tilde{v}) \in \widetilde{E}$, we obtain

$$\iint_{\widetilde{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S}.$$

Opposite of a Parametrized Surface

Let $\Phi : E \to \mathbb{R}^3$ be a smooth parametrized surface, and let $\widetilde{E} := \{(v, u) : (u, v) \in E\}$. Define

$$\widetilde{\Phi}(\tilde{u}, \tilde{v}) := \Phi(\tilde{v}, \tilde{u}) \quad \text{for } (\tilde{u}, \tilde{v}) \in \widetilde{E}.$$

Then the parametrized surface $\widetilde{\Phi}$ is called the **opposite** of Φ, and it will be denoted by Φ^{op}. Clearly, $(\Phi^{\text{op}})^{\text{op}} = \Phi$.

Note that $\Phi^{\text{op}}(\widetilde{E}) = \Phi(E)$.

Also, by the chain rule, for all $(\tilde{u}, \tilde{v}) \in \widetilde{E}$,

$$(\Phi^{\text{op}}_{\tilde{u}} \times \Phi^{\text{op}}_{\tilde{v}})(\tilde{u}, \tilde{v}) = \Phi_v(\tilde{v}, \tilde{u}) \times \Phi_u(\tilde{v}, \tilde{u}) = -(\Phi_u \times \Phi_v)(\tilde{v}, \tilde{u}).$$

Note: If $h(\tilde{u}, \tilde{v}) = (\tilde{v}, \tilde{u})$ for $(\tilde{u}, \tilde{v}) \in \widetilde{E}$, then $J(h)(\tilde{u}, \tilde{v}) = -1$.

If $\boldsymbol{F}$ is a continuous vector field on $\Phi(E)$, then we obtain

$$\iint_{\Phi^{\text{op}}} \boldsymbol{F} \cdot d\boldsymbol{S} = -\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S}.$$