MA 105 : Calculus
Division 3, Lecture 37

Prof. Sudhir R. Ghorpade
IIT Bombay

Recap of the previous lecture

- Oriented surfaces
- Orientation-preserving/reversing parametrizations
- Examples, including Mobius strip and its paramerization
- Intrinsic boundary of a surface
- Orientation of the intrinsic boundary
- The Stokes Theorem
- Sketch of proof of Stokes Theorem
- Vanishing of surface integrals of curl fields as a Corollary
- Surface-independence. Examples

Surface-Independence

Let $E \subset \mathbb{R}^2$, and consider a parametrized surface $\Phi : E \to \mathbb{R}^3$. Let $D \subset \mathbb{R}^3$. We say that Φ lies in D if $\Phi(E) \subset D$.

Let $\boldsymbol{F} : D \to \mathbb{R}^3$ be a continuous vector field. We say that flux integrals of $\boldsymbol{F}$ are **surface-independent** in D if

$$\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{\widetilde{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S}$$

whenever the piecewise C^2 parametrized surfaces $\Phi : E \to \mathbb{R}^3$ and $\widetilde{\Phi} : \widetilde{E} \to \mathbb{R}^3$ lie in D, and if $S := \Phi(E)$, $\widetilde{S} := \widetilde{\Phi}(\widetilde{E})$, then S and $\widetilde{S}$ are oriented (geometric) surfaces and have the same intrinsic boundary which consists of a finite number of nonintersecting simple closed curves and on which the induced orientation is the same. Note that the geometric surfaces S and $\widetilde{S}$ need not be the same.

(Compare the definition of path-independence of a vector field given in Lecture 31.)

Examples: Let $E := [0, \pi/2] \times [-\pi, \pi]$, and let $\Phi(\varphi, \theta) := (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ for $(\varphi, \theta) \in E$. Also, let $\widetilde{E} := \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$, and let $\widetilde{\Phi}(u, v) := (u, v, 0)$ for $(u, v) \in \widetilde{E}$. Then the oriented intrinsic boundaries of $S := \Phi(E)$ and $\widetilde{S} := \widetilde{\Phi}(E)$ are the same, namely $\{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$, traced counterclockwise as seen from $(0, 0, 2)$.

(i) Let $\boldsymbol{F}(x, y, z) := (y, z, x)$ for $(x, y, z) \in \mathbb{R}^3$. Then $\iint_\Phi \boldsymbol{F} \cdot d\boldsymbol{S}$ is equal to

$$\iint_E (\sin\varphi\sin\theta, \cos\varphi, \sin\varphi\cos\theta) \cdot (\sin\varphi)\Phi(\varphi, \theta)d(\varphi, \theta)$$

which is equal to 0 since

$$\int_{-\pi}^{\pi} \sin\theta\cos\theta\, d\theta = \int_{-\pi}^{\pi} \sin\theta\, d\theta = \int_{-\pi}^{\pi} \cos\theta\, d\theta = 0.$$

Also, since $\widetilde{\Phi}_u \times \widetilde{\Phi}_v = (1, 0, 0) \times (0, 1, 0) = (0, 0, 1)$, we obtain

$$\iint_{\widetilde{\Phi}} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{\widetilde{E}} (v,0,u)\cdot(0,0,1)d(u,v) = \iint_{\widetilde{E}} u\,d(u,v)$$
$$= \int_{-\pi}^{\pi}\int_{0}^{1} (r\cos\theta) r\,dr\,d\theta = 0.$$

In fact, we shall presently see that flux integrals of $\boldsymbol{F}$ are surface-independent in $\mathbb{R}^3$.

(ii) Let $\boldsymbol{F}(x,y,z) := (x,y,z)$ for $(x,y,z) \in \mathbb{R}^3$. Then

$$\iint_{\Phi} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{E} \Phi(\varphi,\theta)\cdot(\sin\varphi)\Phi(\varphi,\theta)d(\varphi,\theta) = 2\pi.$$

On the other hand,

$$\iint_{\widetilde{\Phi}} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{\widetilde{E}} (u,v,0)\cdot(0,0,1)d(u,v) = 0.$$

Thus flux integrals of $\boldsymbol{F}$ are not surface-independent in $\mathbb{R}^3$.

Theorem

Let $D \subset \mathbb{R}^3$ be open, and let $\boldsymbol{F}$ be a continuous curl field on D. Then flux integrals of $\boldsymbol{F}$ are surface-independent in D.

Proof: Let $\boldsymbol{G}$ be a smooth vector field on D such that $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$ on D. Let $\Phi : E \to \mathbb{R}^3$ be a piecewise C^2 parametrized surface in D, and let the intrinsic boundary $\eth S$ of the oriented surface $S := \Phi(E)$ consist of a finite number of nonintersecting simple closed curves.

If $\eth S$ is given the induced orientation, then by the Stokes theorem,

$$\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{S} \operatorname{curl} \boldsymbol{G} \cdot d\boldsymbol{S} = \int_{\eth S} \boldsymbol{G} \cdot d\boldsymbol{s},$$

which clearly depends only on $\boldsymbol{G}$, and on $\eth S$ and its orientation, but not on Φ. Hence the result follows. □

If $\boldsymbol{F}$ is a curl field, then the evaluation of a surface integral $\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S}$ can be simplified by replacing Φ by a parametrized surface having the same oriented intrinsic boundary. If we actually know a vector field $\boldsymbol{G}$ such that $\operatorname{curl} \boldsymbol{G} = \boldsymbol{F}$, then we can reduce our problem to the calculation of a line integral.

Example: Let $\boldsymbol{F}(x, y, z) := (y, z, x)$ for $(x, y, z) \in \mathbb{R}^3$.
Consider $\boldsymbol{G}(x, y, z) := (z^2/2, x^2/2, y^2/2)$ for $(x, y, z) \in \mathbb{R}^3$.
Then $\operatorname{curl} \boldsymbol{G} = \boldsymbol{F}$ on $\mathbb{R}^3$, and so flux integrals of $\boldsymbol{F}$ are surface-independent in $\mathbb{R}^3$.
Let $E := [0, \pi/2] \times [-\pi, \pi]$, and for $(\varphi, \theta) \in E$, let $\Phi(\varphi, \theta) := (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi)$. Let $S := \Phi(E)$. Then

$$\begin{aligned} \iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} &= \iint_{S} \operatorname{curl} \boldsymbol{G} \cdot d\boldsymbol{S} = \int_{\partial S} \boldsymbol{G} \cdot d\boldsymbol{s} \\ &= \frac{1}{2} \int_{-\pi}^{\pi} (0, \cos^2 t, \sin^2 t) \cdot (-\sin t, \cos t, 0)\, dt = 0 \end{aligned}$$

since $\cos^3 t = (\cos 3t + 3\cos t)/4$ for $t \in [-\pi, \pi]$.

It is important to know whether a continuous vector field $\boldsymbol{F}$ is in fact a curl field, that is, whether there is a smooth vector field $\boldsymbol{G}$ such that $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$.

We already know a necessary condition when $\boldsymbol{F}$ is smooth:

If $\boldsymbol{F}$ is smooth and $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$ on an open subset D of $\mathbb{R}^3$, then $\operatorname{div}(\boldsymbol{F}) = 0$ on D.

Examples:

(i) Let $D := \mathbb{R}^3$, and $\boldsymbol{F} := (P, Q, R)$, where $P(x, y, z) := x$, $Q(x, y, z) := y$ and $R(x, y, z) := z$ for $(x, y, z) \in D$. Since $\operatorname{div} \boldsymbol{F} = 3 \neq 0$, the vector field $\boldsymbol{F}$ is not a curl field on D.

(ii) Even if $\operatorname{div} \boldsymbol{F} = 0$ on an open subset D of $\mathbb{R}^3$, the vector field $\boldsymbol{F}$ may not be a curl field on D. For instance, let $D := \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$, and let $\boldsymbol{F} := (P, Q, R)$, where for $(x, y, z) \in D$,

$$P := \frac{x}{(x^2+y^2+z^2)^{3/2}}, \quad Q := \frac{y}{(x^2+y^2+z^2)^{3/2}}, \quad R := \frac{z}{(x^2+y^2+z^2)^{3/2}}.$$

Then $(x^2+y^2+z^2)^3 \operatorname{div} \boldsymbol{F} = (x^2+y^2+z^2)^3(P_x+Q_y+R_z) = 3(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{1/2}(x^2+y^2+z^2) = 0.$

Thus $\operatorname{div} \boldsymbol{F} = 0$ on D. But $\boldsymbol{F}$ is not a curl field on D since the standard unit sphere $S := \Phi([0,\pi]\times[-\pi,\pi])$ lies in D, and

$$\begin{aligned} \iint_{\Phi} \boldsymbol{F}\cdot d\boldsymbol{S} &= \iint_{E} \Phi(\varphi,\theta)\cdot(\sin\varphi)\Phi(\varphi,\theta)d(\varphi,\theta) \\ &= \int_{-\pi}^{\pi}\left(\int_{0}^{\pi}\sin\varphi\, d\varphi\right)d\theta = 4\pi \neq 0. \end{aligned}$$

But if $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$, then $\iint_{\Phi} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{S} \operatorname{curl} \boldsymbol{G}\cdot d\boldsymbol{S} = 0$ since $\partial S = \emptyset$. (Recall the corollary of the Stokes theorem.)

If $D := I\times J\times K$, where I, J, K are open intervals in $\mathbb{R}$, and $\boldsymbol{F}$ is a smooth vector field on D such that $\operatorname{div} \boldsymbol{F} = 0$ on D, then $\boldsymbol{F}$ is indeed a curl field. In fact, this also holds if D is an open subset of $\mathbb{R}^3$ such that every closed surface in D is the boundary of a solid lying entirely in D.

Let $D := I \times J \times K$, where I, J, K are open intervals in $\mathbb{R}$. Suppose $\boldsymbol{F} := (P, Q, R)$ is a smooth vector field on D such that $\operatorname{div} \boldsymbol{F} = 0$ on D. We find a vector field $\boldsymbol{G} := (L, M, N)$ on D such that $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$, that is,

$$(P, Q, R) = (N_y - M_z, L_z - N_x, M_x - L_y).$$

Let $x_0 \in I$, $z_0 \in K$.

Define $L := 0$ and $N(x, y, z) := -\int_{x_0}^{x} Q(u, y, z)du$.

Then $L_z - N_x = 0 + Q = Q$. Next, define

$M(x, y, z) := \int_{x_0}^{x} R(u, y, z)du - \int_{z_0}^{z} P(x_0, y, v)dv$. Then

$M_x - L_y = R - 0 = R$.

Finally, by differentiating under the integral sign,

$$\begin{aligned} N_y(x, y, z) &= -\int_{x_0}^{x} Q_y(u, y, z)du = \int_{x_0}^{x} (P_x + R_z)(u, y, z)du \\ &= P(x, y, z) - P(x_0, y, z) + \int_{x_0}^{x} R_z(u, y, z)du, \end{aligned}$$

and

$$M_z(x, y, z) = \int_{x_0}^{x} R_z(u, y, z)du - P(x_0, y, z).$$

Hence $N_y - M_z = P$. Thus $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$.

Further, if there is a vector field $\boldsymbol{H}$ on D such that $\boldsymbol{F} = \operatorname{curl} \boldsymbol{H}$ as well, then $\operatorname{curl}(\boldsymbol{H} - \boldsymbol{G}) = 0$ on D, and so $\boldsymbol{H} = \boldsymbol{G} + \nabla f$, where f is a smooth scalar field on D, as we have mentioned while considering 'path-independence' of vector fields in Lecture 32. This will follow as a consequence of the Stokes theorem since D is simply connected.

Example:

Let $\boldsymbol{F} := (P, Q, R)$, where $P := x$, $Q = -2\,y$ and $R := z$ for $(x, y, z) \in \mathbb{R}^3$. Clearly, $\operatorname{div} \boldsymbol{F} = 0$ on $\mathbb{R}^3$.
Let $x_0 := 0 =: z_0$. Define $L := 0$, $N := 2\,x\,y$ and $M := z\,x$ as above. If we let $\boldsymbol{G} := (0, z\,x, 2\,x\,y)$, then $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$. Also, if $\operatorname{curl} \boldsymbol{H} = \boldsymbol{F}$, then $\boldsymbol{H} = \boldsymbol{G} + (f_x, f_y, f_z)$, where f is a smooth scalar field on $\mathbb{R}^3$.

Let us recall the statement of Stokes' Theorem.

Let S be an oriented (geometric) surface in $\mathbb{R}^3$ which is a closed and bounded subset of $\mathbb{R}^3$, which is piecewise C^2 and whose piecewise smooth intrinsic boundary ∂S consists of a finite number of nonintersecting simple closed curves along with the induced orientation. Let $\boldsymbol{F}$ be a smooth vector field on an open subset containing S. Then

$$\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}.$$

Computational Uses of the Stokes Theorem

(i) Calculation of the flux integral of a curl field

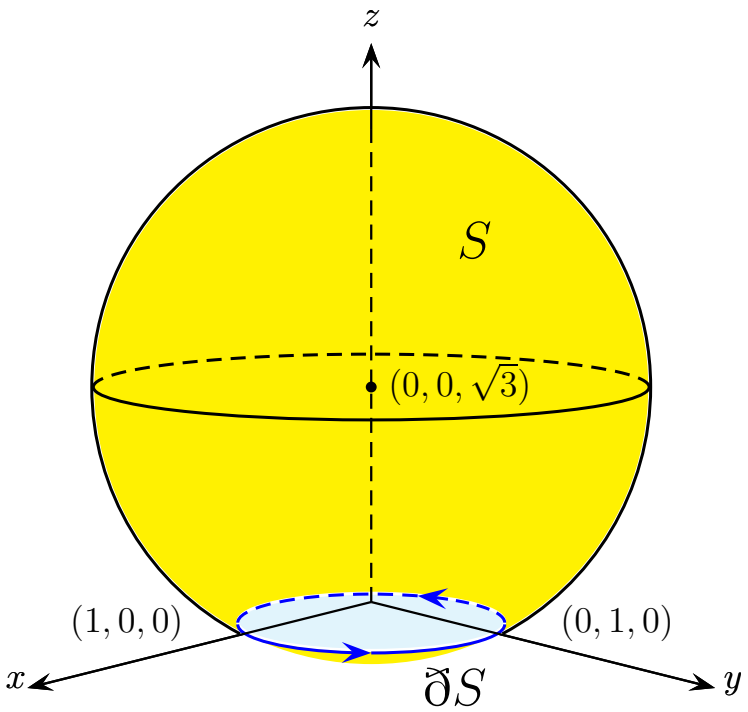
Example:

Let $\boldsymbol{F}(x,y,z) := (y, -x, e^{x\,z})$ for $(x,y,z) \in \mathbb{R}^3$, and let $S := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + (z-\sqrt{3})^2 = 4 \text{ and } z \geq 0\}$, be oriented by the outward unit normal vectors.

In order to find $\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}$, we note that the induced orientation on $\partial S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \text{ and } z = 0\}$ is anticlockwise as seen from the point $(0,0,4)$.

By the Stokes theorem,

$$\begin{aligned}
\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} &= \int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} \\
&= \int_{-\pi}^{\pi} (\sin t, -\cos t, e^{\cos t \cdot 0}) \cdot (-\sin t, \cos t, 0)\, dt \\
&= \int_{-\pi}^{\pi} -(\sin^2 t + \cos^2 t)\, dt = -2\pi.
\end{aligned}$$



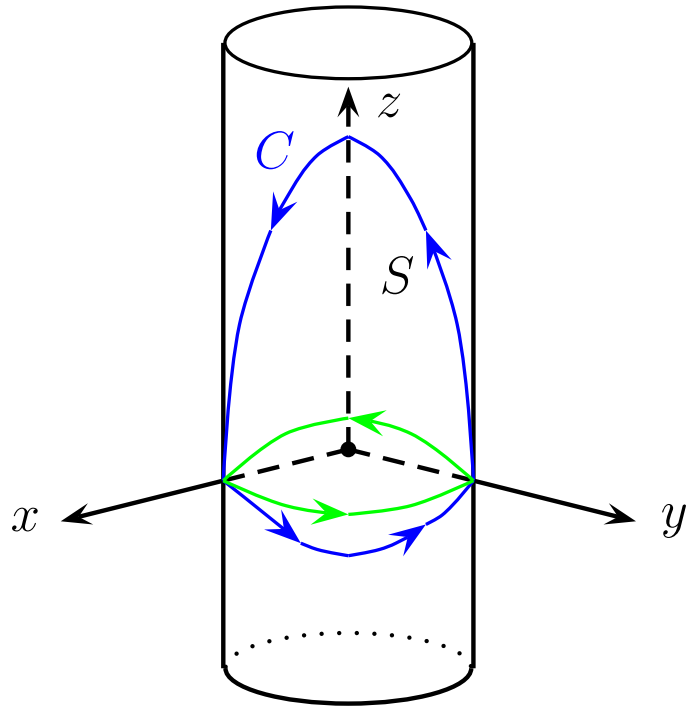


Figure: Examples (i) and (ii) of the use of the Stokes theorem

(ii) Calculation of the line integral along an oriented boundary

Example:
Let $\boldsymbol{F}(x,y,z) := (-y^3, x^3, -z^3)$ for $(x,y,z) \in \mathbb{R}^3$, and let C denote the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$, oriented by the anticlockwise motion on the projection $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ of C on the xy-plane.
In order to find $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$, consider the surface

$$S := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq 1 \text{ and } x+y+z=1\}.$$

Let us think of S as the graph of the function $f : D_0 \to \mathbb{R}$, where $D_0 := \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ and for $(x,y) \in D_0$, $f(x,y) := 1-x-y$. Let S be oriented by the upward unit normal vectors. Then $\eth S = C$, and the induced orientation is the same as the given orientation of C. Hence

$$d\boldsymbol{S} = (-f_x, -f_y, 1)d(x,y) = (1,1,1)d(x,y).$$

Also,

$$\operatorname{curl} \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = (0, 0, 3x^2 + 3y^2),$$

and so $(\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = 3(x^2 + y^2)d(x, y)$.

By the Stokes theorem,

$$\begin{aligned} \int_C \boldsymbol{F} \cdot d\boldsymbol{s} &= \iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = 3 \iint_{D_0} (x^2 + y^2) d(x, y) \\ &= 3 \int_0^1 \int_{-\pi}^{\pi} r^2 r \, dr \, d\theta = \frac{3\pi}{2}. \end{aligned}$$

Consequences of the Stokes Theorem

Proposition

Let $\boldsymbol{F}$ be a smooth vector field on an open subset D of $\mathbb{R}^3$ such that $\operatorname{curl} \boldsymbol{F} = \boldsymbol{0}$ on D.
(i) Suppose S is a bounded oriented piecewise C^2 surface in D, and let ∂S denote its intrinsic boundary with the induced orientation, as in the Stokes theorem. Then $\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} = 0$.

In particular, if $\partial S = C_1 \cup (-C_2)$, so that C_1 and $-C_2$ have the induced orientation, then $\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{s}$.

(ii) If D is simply connected, then $\boldsymbol{F}$ is a gradient field on D.

Proof:
(i) By the Stokes theorem,

$$\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = \iint_S \boldsymbol{0} \cdot d\boldsymbol{S} = 0.$$