MA 105 : Calculus
Division 3, Lecture 04

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Recap of the previous lecture

- Quick review of the previous lecture
- Example of a divergent sequence
- Basic results:
  - Uniqueness of limit
  - Convergent sequences are bounded
  - Limit theorems for sequences
  - Sandwich theorem
- Application to the sequence of powers of a fixed real number

Announcement: *My office hours today will be 5.30 pm to 7 pm instead of 10.30 – 11.30 am, The change is only for this week.*

Monotonic sequences

Definition

A sequence (a_n) is said to be **increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, that is, $a_1 \leq a_2 \leq a_3 \leq \cdots$.

A sequence (a_n) is said to be **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$, that is, $a_1 \geq a_2 \geq a_3 \geq \cdots$.

A sequence is **monotonic** if it is either increasing or decreasing.

Examples:

- The sequences (n) and $(-1/n)$ are increasing.
- The sequences $(-n)$ and $(1/n)$ are decreasing.
- A constant sequence is both increasing and decreasing.
- The sequences $((-1)^n)$ and $((-1)^n n)$ are neither increasing nor decreasing. Thus they are not monotonic.

Convergence theorems

- An increasing sequence (a_n) that is bounded above is convergent, and $\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$.
- A decreasing sequence (a_n) that is bounded below is convergent, and $\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$.

Thus, a monotonic bounded sequence is convergent.

Proof: Suppose (a_n) is increasing and bounded above. Let $a := \sup\{a_n : n \in \mathbb{N}\}$. Let $\epsilon > 0$. Then $a - \epsilon$ is not an upper bound of $\{a_n : n \in \mathbb{N}\}$, and so there is $n_0 \in \mathbb{N}$ such that $a - \epsilon < a_{n_0}$. But then

$$a - \epsilon < a_{n_0} \leq a_n \leq a \quad \text{for all } n \geq n_0.$$

Thus $|a_n - a| < \epsilon$ for all $n \geq n_0$. Hence $a_n \to a$. A similar proof holds for a decreasing sequence that is bounded below.

Examples: (i) Let $a_n := 1 + (1/2^2) + \cdots + (1/n^2)$ for $n \in \mathbb{N}$. The sequence (a_n) is clearly increasing. Further, since

$$\begin{aligned} a_n &\leq 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1) \cdot n} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \end{aligned}$$

we see that $a_n \leq 2 - (1/n) < 2$. Thus (a_n) is bounded. Hence by the result in the previous slide, (a_n) is convergent.

(ii) Consider the sequence (a_n) defined recursively by

$$a_1 := \frac{3}{2} \quad \text{and} \quad a_{n+1} := \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad \text{for } n \in \mathbb{N}.$$

Note that if $1 \leq a_n \leq 2$, then it is easily deduced from the recurrence relation that $1 \leq a_{n+1} \leq 2$. Since $1 \leq a_1 \leq 2$, we see by induction that $1 \leq a_n \leq 2$ for all $n \in \mathbb{N}$. In particular, (a_n) is bounded. Also, $a_2 = 17/12 < a_1$.

Let us check whether the sequence (a_n) is decreasing. Since

$$a_n - a_{n+1} = a_n - \frac{a_n^2 + 2}{2a_n} = \frac{a_n^2 - 2}{2\,a_n} \quad \text{for all } n \in \mathbb{N},$$

(a_n) is decreasing if and only if $a_n^2 - 2 \geq 0$ for all $n \in \mathbb{N}$. But

$$a_1^2 \geq 2 \quad \text{and} \quad a_{n+1}^2 - 2 = \frac{(a_n^2 - 2)^2}{4a_n^2} \geq 0 \text{ for all } n \in \mathbb{N}.$$

Hence the sequence (a_n) is decreasing. It follows that (a_n) is convergent. Let $a_n \to a$. Then $a_{n+1} \to a$ as well. Moreover, since $1 \leq a_n \leq 2$ for all $n \in \mathbb{N}$, we see that $1 \leq a \leq 2$. Now

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \to \frac{1}{2}\left(a + \frac{2}{a}\right).$$

Since the limit of a sequence is unique, we see that

$$\frac{1}{2}\left(a + \frac{2}{a}\right) = a, \quad \text{that is,} \quad a^2 = 2.$$

Thus $a = \pm\sqrt{2}$. But since $1 \leq a \leq 2$, it follows that $a = \sqrt{2}$.

Continuity of a function of a real variable

Let D be a subset of $\mathbb{R}$, and f be a real-valued function defined on D, that is, let $f : D \to \mathbb{R}$. Recall that the subset $f(D) := \{f(x) : x \in D\}$ of $\mathbb{R}$ is called the **range** of f.

Definition

Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$ and $c \in D$. We say that f is **continuous** at c if
(x_n) is a sequence in D, $x_n \to c \implies f(x_n) \to f(c)$.

We say that f is **continuous on** D if f is continuous at each point of D.

We say that f is **discontinuous** on D if it is not continuous on D, that is, if there is $c \in D$ at which f is not continuous.

Examples: (i) Let $D \subseteq \mathbb{R}$. A constant function from D to $\mathbb{R}$ is clearly continuous. Also $f : D \to \mathbb{R}$ defined by $f(x) := x$ for $x \in D$, is clearly continuous on D.

(ii) Let $D := [0, \infty)$ and $f(x) := \sqrt{x}$ for $x \in [0, \infty)$. First note that if $c_n \in [0, \infty)$ for $n \in \mathbb{N}$ and $c_n \to 0$, then $\sqrt{c_n} \to 0$. (Pf: Given $\epsilon > 0$, choose $n_0 \in \mathbb{N}$ such that $|c_n| < \epsilon^2$ for all $n \geq n_0$. Then $|\sqrt{c_n}| < \epsilon$ for all $n \geq n_0$.) Now note that

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|} \quad \text{for all } x, y \in [0, \infty).$$

(Pf: $x = \left(\sqrt{x} - \sqrt{y} + \sqrt{y}\right)^2 \geq \left(\sqrt{x} - \sqrt{y}\right)^2 + y$. Hence etc.)

Let $c \in (0, \infty)$ and let $x_n \to c$, where $x_n \geq 0$ for each $n \in \mathbb{N}$. Then $|x_n - c| \to 0$, and so $|\sqrt{x_n} - \sqrt{c}| \leq \sqrt{|x_n - c|} \to 0$, which implies that $f(x_n) \to f(c)$. Thus f is continuous at c.

More generally, if $k \in \mathbb{N}$, and $f(x) := x^{1/k}$ for $x \in [0, \infty)$, then f is continuous on $[0, \infty)$. This follows from the Optional Exercise mentioned in Lecture 1. Another proof of this result will be given in a later lecture.

Basic Properties of Continuous functions

- Let $D \subset \mathbb{R}$, and let $f, g : D \to \mathbb{R}$ be functions. Let $c \in D$, and let f and g be continuous at c. Then $f + g$ and $f \cdot g$ are continuous at c. Further, f/g is continuous at c if $g(c) \neq 0$. These results follow from the limit theorems for sequences.

  Consequences:

  (i) A polynomial is continuous on $\mathbb{R}$.

  (ii) A **rational function** $r(x) := p(x)/q(x)$, where p and q are polynomials, is continuous at $c \in \mathbb{R}$ if $q(c) \neq 0$.

- Composite of continuous functions: Let $D \subset \mathbb{R}$, $E \subset \mathbb{R}$. If $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are functions such that $f(D) \subseteq E$, f is continuous at $c \in D$, and g is continuous at $f(c) \in E$, then $g \circ f : D \to \mathbb{R}$ is continuous at c.

Equivalent condition for continuity of a function

Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$ and $c \in D$. Then f is continuous at c if and only if the following ϵ-δ condition holds:
For every $\epsilon > 0$, there exists $\delta > 0$ such that

$$x \in D \text{ and } |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

Note: Proof is skipped. (For those interested, deriving the continuity from the ϵ-δ condition is easy. The converse isn't difficult if you write down the negation of the ϵ-δ condition.)

Example: Define $f(x) := \begin{cases} x \sin(1/x) & \text{if } x \in \mathbb{R},\ x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

Given any $\epsilon > 0$, we can take $\delta := \epsilon$. Then

$$x \in \mathbb{R}, \text{ and } |x - 0| < \delta \implies |f(x) - f(0)| \leq |x| < \delta = \epsilon.$$

Hence f is continuous at 0.

Intermediate value property of a function

Definition

Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to have the **intermediate value property** (IVP) on I if for any $a, b \in I$ with $a < b$, and $r \in \mathbb{R}$,

r lies between $f(a)$ and $f(b) \implies r = f(c)$ for some $c \in [a, b]$.

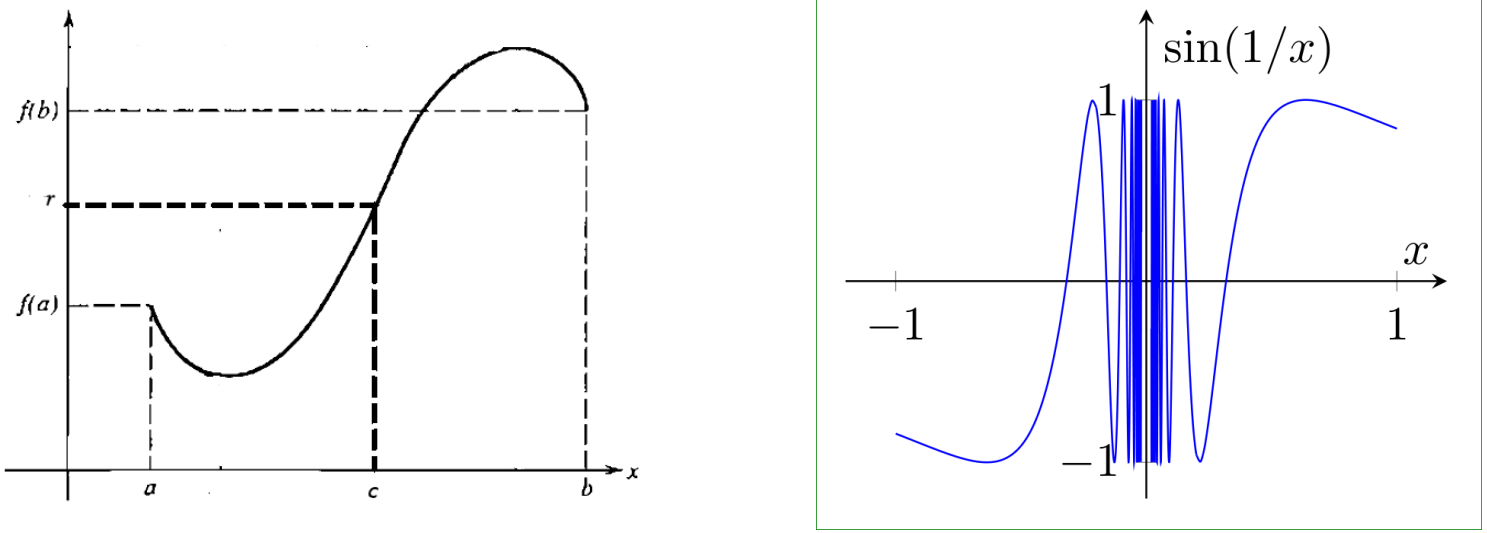


Figure: Illustration of (i) a continuous function having the IVP, and (ii) a discontinuous function having the IVP

Theorem (Intermediate value theorem)

Let I be an interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a continuous function. Then f has the intermediate value property on I. In particular, the range $J := f(I)$ of f is an interval.

Proof (Sketch): Let $a, b \in I$ with $a < b$. First, suppse $r \in (f(a), f(b))$, Consider $S := \{x \in [a, b] : f(x) < r\}$ and $c := \sup S$. Find sequence (c_n) in S with $c_n \to c$. Use the continuity of f to obtain $f(c) \leq r$ and $a \leq c < b$. Also find (d_n) in (c, b) with $d_n \to c$ to conclude $a < c$ and $f(c) \geq r$.

Examples: (i) Define $f : [0, 1] \to \mathbb{R}$ by $f(0) := 0$ and $f(x) := \sin(1/x)$ if $0 < x \leq 1$. Then f has the IVP. But f is not continuous at 0. So the converse of above theorem is false.

(ii) Define $f : [-1, 2] \to \mathbb{R}$ by $f(x) := 1$ if $x < 0$, $f(0) := 2$ and $f(x) := x$ if $x > 0$. Now $f(-1) = 1 < 1.5 < 2 = f(0)$. But $1.5 \neq f(x)$ for any $x \in [-1, 0]$ (although $f(1.5) = 1.5$). Thus f does not have the IVP. Note: $f([-1, 2]) = (0, 2]$.