MA 105 : Calculus
Division 3, Lecture 05

Prof. Sudhir R. Ghorpade
IIT Bombay

Recap of the previous lecture

- Monotonic sequences and their basic properties. Examples
- Oral mention of supplementary topics: Notion of subsequences and Cauchy sequences. Bolzano-Weierstrass Theorem and Cauchy Criterion
- Continuity of a function at a point (definition using sequences). Continuous and discontinuous functions. Examples
- Basic properties: Sums, scalar multiples, products, quotients and composites of continuous functions are continuous wherever defined.
- Consequences, including continuity of rational functions wherever they are defined
- Equivalent definition of continuity (ϵ-δ condition)
- Intermediate Value Property (IVP). Examples
- Intermediate Value Theorem.

Intermediate value property of a function

Definition

Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to have the **intermediate value property** (IVP) on I if for any $a, b \in I$ with $a < b$, and $r \in \mathbb{R}$,

r lies between $f(a)$ and $f(b) \implies r = f(c)$ for some $c \in [a, b]$.

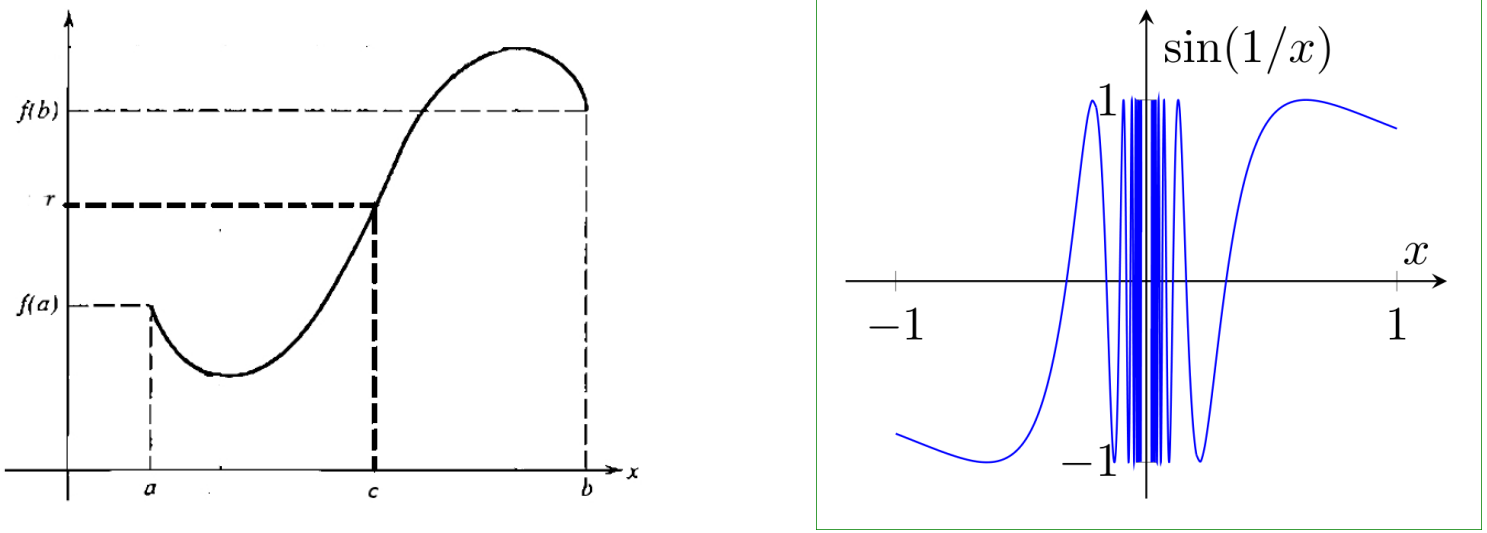

Figure: Illustration of (i) a continuous function having the IVP, and (ii) a discontinuous function having the IVP

Theorem (Intermediate value theorem)

Let I be an interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a continuous function. Then f has the intermediate value property on I. In particular, the range $J := f(I)$ of f is an interval.

Proof (Sketch): Let $a, b \in I$ with $a < b$. First, suppse $r \in (f(a), f(b))$, Consider $S := \{x \in [a, b] : f(x) < r\}$ and $c := \sup S$. Find sequence (c_n) in S with $c_n \to c$. Use the continuity of f to obtain $f(c) \leq r$ and $a \leq c < b$. Also find (d_n) in (c, b) with $d_n \to c$ to conclude $a < c$ and $f(c) \geq r$.

Examples: (i) Define $f : [0, 1] \to \mathbb{R}$ by $f(0) := 0$ and $f(x) := \sin(1/x)$ if $0 < x \leq 1$. Then f has the IVP. But f is not continuous at 0. So the converse of above theorem is false.

(ii) Define $f : [-1, 2] \to \mathbb{R}$ by $f(x) := 1$ if $x < 0$, $f(0) := 2$ and $f(x) := x$ if $x > 0$. Now $f(-1) = 1 < 1.5 < 2 = f(0)$. But $1.5 \neq f(x)$ for any $x \in [-1, 0]$ (although $f(1.5) = 1.5$). Thus f does not have the IVP. Note: $f([-1, 2]) = (0, 2]$.

An application of the Intermediate Value Theorem

(i) Let $n \in \mathbb{N}$ be odd, and let $p(x) := a_0 + a_1 x + \cdots + a_n x^n$ for $x \in \mathbb{R}$, where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ and $a_n \neq 0$. Then p has at least one (real) root.

Idea behind a proof:

- Suppose $a_n > 0$ without loss of generality.
- If we take $x := b > 0$ large enough, then $p(b) > 0$.
- If we take $x := a < 0$ small enough, then $p(a) < 0$.
- Since p is continuous, it has the IVP on $\mathbb{R}$.
- The number 0 lies between $p(a)$ and $p(b)$.
- There is $c \in \mathbb{R}$ such that $a < c < b$ and $p(c) = 0$.

(ii) The polynomial $p(x) = x^4 + 2x^3 - 2$ has a root in $(0, 1)$: Since p is continuous on [0,1], it has the IVP on $[0, 1]$. Also, $p(0) = -2 < 0$, and $p(1) = 1 > 0$, and so there is $c \in (0, 1)$ such that $p(c) = 0$.

Continuous functions on $[a, b]$

Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is called **bounded** if there is $\alpha \in \mathbb{R}$ such that $|f(x)| \leq \alpha$ for all $x \in D$. We say that a function f **attains its bounds** if there are $x_1, x_2 \in D$ with $f(x_1) = \inf\{f(x) : x \in D\}$ and $f(x_2) = \sup\{f(x) : x \in D\}$.

Theorem (Extreme value property)

Let $[a, b]$ be a closed and bounded interval, and let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is bounded and attains its bounds on $[a, b]$. Thus there are $c_1, c_2 \in [a, b]$ such that

$$\begin{aligned} f(c_1) &= \min\{f(x) : x \in [a, b]\}, \\ f(c_2) &= \max\{f(x) : x \in [a, b]\}. \end{aligned}$$

In fact, $f([a, b])$=$[f(c_1), f(c_2)]$, a closed and bounded interval.

Proof: Skipped. See [GL-1] if you are interested.

Examples:

(i) Define $f : (0, 1] \to \mathbb{R}$ by $f(x) := 1/x$ for $x \in (0, 1]$.
Then f is continuous but unbounded.

(ii) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := x$ for $x \in \mathbb{R}$.
Then f is continuous but unbounded.

(iii) Define $f : [-1, 1] \to \mathbb{R}$ by $f(x) := 1/x$ if $x \neq 0$ and $f(0) := 0$. Then f is discontinuous and unbounded.

(iv) Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) := 1/(1 + x^2)$.
Then f is continuous and bounded, but it does not attain its bounds. (Note: $\sup\{f(x) : x \in (0, \infty)\} = 1$ and $\inf\{f(x) : x \in (0, \infty)\} = 0$.)

(v) Define $f : [-1, 1] \to \mathbb{R}$ by $f(x) = -1$ if $x < 0$ and $f(x) := 1$ if $x \geq 0$. Then f is bounded and attains its bounds, but f is not continuous on $[-1, 1]$. Thus the converse of the above theorem is false.

Limits of functions of a real variable

Let $D \subset \mathbb{R}$, and let $c \in \mathbb{R}$. Let $f : D \to \mathbb{R}$, and let $\ell \in \mathbb{R}$.

Suppose the values of the function f are as close to the real number ℓ as we like for all x close enough to c (on either side of c), except possibly when $x = c$, then we say that the limit of $f(x)$, as x approaches c, is equal to $\ell \in \mathbb{R}$.

To consider a limit of $f(x)$ as x approaches c, the function f must be defined near the point c (but not necessarily at c). To do this, we may require that D contains an interval of the form $(c - r, c + r)$, except possibly the point c, for some $r > 0$.

Definition

Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$ and let $c \in \mathbb{R}$ be such that there is $r > 0$ with $(c - r, c) \cup (c, c + r) \subset D$.

We say that $\lim\limits_{x \to c} f(x)$ exists if there is $\ell \in \mathbb{R}$ such that (x_n) is a sequence in D, $x_n \neq c$ and $x_n \to c \implies f(x_n) \to \ell$.

Remark: The condition $(c-r, c) \cup (c, c+r) \subset D$ ensures that there do exist sequences (x_n) in D such that $x_n \neq c$ for all $n \in \mathbb{N}$ and $x_n \to c$. This holds also if we put a *weaker* condition that c is a *limit point* of D, which means for every $r > 0$, there exists $x \in (c-r, c+r)$ with $x \neq c$.

Examples:

(i) Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 3x - 1 & \text{if} \quad x \in \mathbb{R} \text{ and } x \neq 0, \\ 1 & \text{if} \quad x := 0. \end{cases}$$

Let $x_n \to 0$ and $x_n \neq 0$ for $n \in \mathbb{N}$. Then $f(x_n) = 3x_n - 1 \to -1$. Hence $\lim\limits_{x \to 0} f(x) = -1$.

(ii) Let $f(x) := a_0 + a_1 x + \cdots + a_n x^n$ for $x \in \mathbb{R}$, where $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{R}$.

Let $c \in \mathbb{R}$, and let $x_n \to c$, $x_n \neq c$ for all $n \in \mathbb{N}$. Then $f(x_n) \to f(c)$. Hence $\lim\limits_{x \to c} f(x) = \ell$, where $\ell := f(c)$.

Examples (continued)

(iii) Let $f(x) := [x]$ for $x \in \mathbb{R}$.

If $x_n := 1 - (1/n)$ for $n \in \mathbb{N}$, then $x_n \to 1$ and $f(x_n) = 0$ for all $n \in \mathbb{N}$, and so $f(x_n) \to 0$.

If $x_n = 1 + (1/n)$ for $n \in \mathbb{N}$, then $x_n \to 1$ and $f(x_n) = 1$ for all $n \in \mathbb{N}$, and so $f(x_n) \to 1$.

Thus $\lim_{x \to 1} f(x)$ does not exist.

(iv) Let $f(x) := \sin(1/x)$ for $x \in \mathbb{R} \setminus \{0\}$.

If $x_n := 2/(2n+1)\pi$ for $n \in \mathbb{N}$, then $x_n \to 0$, but since $f(x_n) = \sin((2n+1)\pi/2) = (-1)^n$ for all $n \in \mathbb{N}$, we see that $(f(x_n))$ does not converge.

Thus $\lim_{x \to 0} f(x)$ does not exist.

Continuity and Limit

Let us relate the concepts of continuity and limit.

Definition

Let $D \subset \mathbb{R}$. We say that $c \in \mathbb{R}$ is an **interior point** of D if there is $r > 0$ such that $(c - r, c + r) \subset D$.

Theorem

Let c be an interior point of D, and let $f : D \to \mathbb{R}$. Then f is continuous at c $\iff$ $\lim_{x \to c} f(x)$ exists and it is equal to $f(c)$.

Thus for f to be continuous at an interior point c of D,

(i) f must be defined at c,

(ii) $\lim_{x \to c} f(x)$ must exist, and

(iii) $\lim_{x \to c} f(x) = f(c)$.

Limit theorems for functions of a real variable

Suppose $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then

- $\lim_{x\to c}(f \pm g)(x) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$ resp.,
- $\lim_{x\to c}(f \cdot g)(x) = \big(\lim_{x\to c} f(x)\big) \cdot \big(\lim_{x\to c} g(x)\big)$.
- If $\lim_{x\to c} g(x) \neq 0$, then $\lim_{x\to c} \left(\frac{f}{g}\right)(x) = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$.
- If $f(x) \leq g(x)$ for all $x \neq c$ near c, then $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$.
- (Sandwich theorem)
  If $f(x) \leq h(x) \leq g(x)$ for all x near c and $x \neq c$, and if $\lim_{x\to c} f(x) = \ell = \lim_{x\to c} g(x)$, then $\lim_{x\to c} h(x) = \ell$.

The proofs follow from the limit theorems for sequences.

Examples

(i) Let $0 < |x| < \pi/2$. As we shall see later,

$$-|x| < \sin x < |x| \quad \text{and} \quad -|x| < 1 - \cos x < |x|.$$

Hence $\lim_{x \to 0} \sin x = 0$ and $\lim_{x \to 0} \cos x = 1$ by the sandwich theorem.

(ii) Let $0 < x < \pi/2$. As we shall see later,
$0 < x \cos x < \sin x < x$, and $\sin(-x) = -\sin x$ fo $x \in \mathbb{R}$.
Hence $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$ by the sandwich theorem.

(iii) Let $f(x) := x \sin\left(\dfrac{1}{x}\right)$ for $x \in \mathbb{R} \setminus \{0\}$.

Then $-|x| \leq x \sin\left(\dfrac{1}{x}\right) \leq |x|$ for all $x \in \mathbb{R} \setminus \{0\}$.

Hence $\lim_{x \to 0} x \sin\left(\dfrac{1}{x}\right) = 0$ by the sandwich theorem.

Right hand and left hand limits

Definition

Suppose $f : D \to \mathbb{R}$ and $c \in \mathbb{R}$ are such that $(c, c + r) \subset D$ for some $r > 0$. We say that $\lim\limits_{x \to c^+} f(x)$ exists if there is $\ell_1 \in \mathbb{R}$ such that

(x_n) a sequence in D, $x_n \to c$ and $x_n > c \implies f(x_n) \to \ell_1$.

Definition

Suppose $f : D \to \mathbb{R}$ and $c \in \mathbb{R}$ are such that $(c - r, c) \subset D$ for some $r > 0$. We say that $\lim\limits_{x \to c^-} f(x)$ exists if there is $\ell_2 \in \mathbb{R}$ such that

(x_n) a sequence in D, $x_n \to c$ and $x_n < c \implies f(x_n) \to \ell_2$.

Remark: The right hand limit can also be defined more generally if c is a limit point of $D \cap (c, \infty)$. Similarly, the left hand limit can be defined if c is a limit point of $D \cap (-\infty, c)$.

Example: Let $f(x) := [x]$ for $x \in \mathbb{R}$. Then

$$\lim_{x \to 1^-} f(x) = 0 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = 1.$$

Theorem

Let $f : D \to \mathbb{R}$ and let $c \in \mathbb{R}$ such that there is $r > 0$ with

$$(c - r, c) \cup (c, c + r) \subset D.$$

$\lim_{x \to c} f(x)$ exists $\iff$ $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ exist and are equal. In this case,

$$\lim_{x \to c^-} f(x) = \lim_{x \to c} f(x) = \lim_{x \to c^+} f(x).$$

Remark: The above theorem holds more generally if instead of assuming that $(c - r, c) \cup (c, c + r) \subset D$ for some $r > 0$, we make a weaker assumption that c is a limit point of $D \cap (c, \infty)$ and also a limit point of $D \cap (-\infty, c)$.

Equivalent condition for the existence of a limit

Recall:

(Continuity)

Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$ and $c \in D$. Then f is continuous at c $\iff$ the following ϵ-δ condition holds:
For every $\epsilon > 0$, there exists $\delta > 0$ such that

$$x \in D \text{ and } |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

Similarly,

(Limit)

Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$ and $c \in \mathbb{R}$ be such that there is $r > 0$ with $(c - r, c) \cup (c, c + r) \subset D$. Let $\ell \in \mathbb{R}$. Then
$\lim_{x \to c} f(x) = \ell \iff$ the following ϵ-δ condition holds:
For every $\epsilon > 0$, there is $\delta > 0$ such that

$$x \in D \text{ and } 0 < |x - c| < \delta \implies |f(x) - \ell| < \epsilon.$$

Example: Let

$$f(x) := \frac{x}{|x|} \quad \text{for } x \in \mathbb{R} \setminus \{0\}.$$

Then there is no $\ell \in \mathbb{R}$ such that $\lim_{x \to 0} f(x) = \ell$.

For $n \in \mathbb{N}$, let $x_n := (-1)^n/n$. Then $x_n \to 0$, but $(f(x_n))$ is divergent, since $f(x_n) = (-1)^n$ for $n \in \mathbb{N}$. Hence the result.

Aliter: Suppose there is $\ell \in \mathbb{R}$ such that $\lim_{x \to 0} f(x) = \ell$. Let $\epsilon := 1/2$. Then there is $\delta > 0$ such that $|f(x) - \ell| < 1/2$ for all $x \in \mathbb{R}$ satisfying $0 < |x| < \delta$.
Let $x_0 := \delta/2$. Then $f(x_0) = 1$ and $f(-x_0) = -1$. Hence

$$\begin{aligned} 2 &= |f(x_0) - f(-x_0)| \le |f(x_0) - \ell| + |\ell - f(-x_0)| \\ &< \frac{1}{2} + \frac{1}{2} = 1, \quad \text{which is a contradiction.} \end{aligned}$$

Differentiability

Definition

Let $D \subset \mathbb{R}$, and let c be an interior point of D. A function $f : D \to \mathbb{R}$ is said to be **differentiable** at c if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. In this case, this limit is denoted by $f'(c)$, and it is called the **derivative** of f at c.

Geometrically speaking, $f'(c)$ is equal to the slope of the tangent to the curve $y = f(x)$ at $x = c$, and so the curve is 'smooth' at c; there is no corner or cusp at c.

Also, $f'(c)$ can be interpreted as the rate of change in f at c.

Examples:

(i) If $f : \mathbb{R} \to \mathbb{R}$ is a **constant function**, then f is differentiable and $f'(c) = 0$ for all $c \in \mathbb{R}$.

(ii) If $f : \mathbb{R} \to \mathbb{R}$ is the **identity function** $f(x) := x$ for $x \in \mathbb{R}$, then f is differentiable and $f'(c) = 1$ for all $c \in \mathbb{R}$.

(iii) Let $f(x) := x^{2/3}$ for $x \in \mathbb{R}$. Then f is not differentiable at 0 since $\dfrac{f(0+h) - f(0)}{h} = \dfrac{1}{h^{1/3}}$ for $h \neq 0$.

(iv) Let $f(x) := \sin x$ for $x \in \mathbb{R}$. Then f is differentiable at 0 since $\lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{\sin h - \sin 0}{h - 0} = 1.$

(v) Let $f(0) := 0$ and $f(x) := x \sin(1/x)$ for $x \in \mathbb{R} \setminus \{0\}$. Then f is not differentiable at 0 since

$$\frac{f(0+h) - f(0)}{h} = \sin\left(\frac{1}{h}\right) \quad \text{for } h \neq 0.$$