MA 105 : Calculus
Division 3, Lecture 06

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Recap of the previous lecture

- An application of the Intermediate Value Theorem
- Extreme Value Property (of continuous functions defined on closed and bounded intervals), Examples
- Limits of functions of a real variable. Examples
- Continuity and Limit
- Limit theorems for functions of a real variable
- Right hand and left hand limits
- Equivalent condition for the existence of a limit
- Differentiability: Definition and Examples

Differentiability

Definition

Let $D \subseteq \mathbb{R}$, and let c be an interior point of D. A function $f : D \to \mathbb{R}$ is said to be **differentiable** at c if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. In this case, this limit is denoted by $f'(c)$, and it is called the **derivative** of f at c.

Geometrically speaking, $f'(c)$ is equal to the slope of the tangent to the curve $y = f(x)$ at $x = c$, and so the curve is 'smooth' at c; there is no corner or cusp at c.

Also, $f'(c)$ can be interpreted as the rate of change in f at c.

Examples:

(i) If $f : \mathbb{R} \to \mathbb{R}$ is a **constant function**, then f is differentiable and $f'(c) = 0$ for all $c \in \mathbb{R}$.

(ii) If $f : \mathbb{R} \to \mathbb{R}$ is the **identity function** $f(x) := x$ for $x \in \mathbb{R}$, then f is differentiable and $f'(c) = 1$ for all $c \in \mathbb{R}$.

(iii) Let $f(x) := x^{2/3}$ for $x \in \mathbb{R}$. Then f is not differentiable at 0 since $\dfrac{f(0+h) - f(0)}{h} = \dfrac{1}{h^{1/3}}$ for $h \neq 0$.

(iv) Let $f(x) := \sin x$ for $x \in \mathbb{R}$. Then f is differentiable at 0 since $\lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{\sin h - \sin 0}{h - 0} = 1.$

(v) Let $f(0) := 0$ and $f(x) := x \sin(1/x)$ for $x \in \mathbb{R} \setminus \{0\}$. Then f is not differentiable at 0 since

$$\frac{f(0+h) - f(0)}{h} = \sin\left(\frac{1}{h}\right) \quad \text{for } h \neq 0.$$

Right hand and left hand derivatives

Let $D \subset \mathbb{R}$, and let $f : D \to \mathbb{R}$.

- Suppose $c \in D$ is such that $[c, c + r) \subseteq D$ for some $r > 0$. If the limit

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

  exists, then it is called the **right hand derivative** of f at c, and we denote it by $f'_+(c)$.
- The **left hand derivative** of f at c is defined similarly, and we denote it by $f'_-(c)$.
- If c is an interior point of D, then f is differentiable at $c \iff f'_+(c)$ and $f'_-(c)$ both exist and are equal.

Example: Let $f(x) := |x|$ for $x \in \mathbb{R}$. Then $f'_-(0) = -1$ and $f'_+(0) = 1$. Hence f is not differentiable at 0.

- If I is a nonempty open interval in $\mathbb{R}$, that is, if $I := (a, b)$, (a, ∞), $(-\infty, b)$ or $\mathbb{R}$, then we say that a function $f : I \to \mathbb{R}$ is **differentiable on** I if f is differentiable at every $c \in I$.

- If $I := [a, b]$, then we say that $f : I \to \mathbb{R}$ is **differentiable on** I if f is differentiable on (a, b), and if both $f'_+(a)$ and $f'_-(b)$ exist.

- Similarly, we define the differentiability of a function on $I := [a, b)$, $(a, b]$, $[a, \infty)$ or $(-\infty, b]$.

In these cases, there is a new function $f' : I \to \mathbb{R}$, called the **derivative (function)** of f defined on I.
Let $c \in I$. If $g := f'$ is differentiable at c, then we say that f is **twice differentiable** at c, and write $f''(c) := g'(c)$.

$f''(c)$ is called the **second derivative** of f at c. Also, for $n \in \mathbb{N}$, we denote the n**th derivative** of f at c by $f^{(n)}(c)$.

Equivalent condition for differentiability

Let $D \subseteq \mathbb{R}$, let $f : D \to \mathbb{R}$, and let c be an interior point of D.

Recall that f is said to be differentiable at c if the limit

$$f'(c) := \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \quad \text{exists.}$$

(Carathéodory Lemma: C-lemma)

Let $f : D \to \mathbb{R}$, and c be an interior point of D. Then f is differentiable at $c \iff$ there is a function $f_1 : D \to \mathbb{R}$ which is continuous at c and satisfies

$$f(x) - f(c) = (x - c) f_1(x) \quad \text{for all } x \in D.$$

In this case, the function f_1 is unique and $f'(c) = f_1(c)$.

The function $f_1 : D \to \mathbb{R}$ is called the **increment function** associated with f and c.

Proof: Suppose f is differentiable at c. Define

$$f_1(x) := \begin{cases} \dfrac{f(x) - f(c)}{x - c} & \text{if } x \in D \setminus \{c\}, \\ f'(c) & \text{if } x = c. \end{cases}$$

Then f_1 is continuous at c since $\lim_{x \to c} f_1(x) = f'(c) = f_1(c)$.

Conversely, suppose there is $f_1 : D \to \mathbb{R}$ as stated. Let $h := x - c$ for $x \in D$, and so $c + h = x$. Then

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} f_1(x) = f_1(c),$$

since f_1 is continuous at c. Hence f is differentiable at c. The uniqueness of the increment function f_1 is obvious. □

If D is an interval and c is an end-point of D, then the C-lemma is valid at c.

The C-lemma relates the concepts of differentiability and continuity. It enables us to prove the following results neatly.

Differentiability $\Longrightarrow$ Continuity

Theorem

Suppose $f : D \to \mathbb{R}$ is differentiable at an interior point c of D. Then f is continuous at c.

Proof.

Let f_1 be the increment function associated with f and c. Then $f(x) = f(c) + (x - c)f_1(x)$ for $x \in I$. Since $f_1 : I \to \mathbb{R}$ is continuous at c, so is f. $\square$

Remarks:

1. If a function f is not continuous at an interior point c of D, then it cannot be differentiable at c.
Example: Let $f(x) := [x]$ for $x \in \mathbb{R}$, and $c := 1$.

2. The converse of the above theorem is false.
Example: $f(x) = |x|$ for $x \in \mathbb{R}$, and $c := 0$.

Differentiability: Algebraic rules

Let f and g be differentiable at c. Then $f \pm g$ and $f \cdot g$ are differentiable at c, and

$$(f \pm g)'(c) = f'(c) \pm g'(c) \quad \text{resp.,}$$

$$(f \cdot g)'(c) = f'(c)g(c) + f(c)g'(c).$$

Further, if $g(c) \neq 0$, then f/g is differentiable at c, and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.$$

Note: If f is differentiable at c, then f^n is differentiable at c for each $n \in \mathbb{N}$, and $(f^n)'(c) = n\, f(c)^{n-1} f'(c)$. This also holds for a negative integer n if $f(c) \neq 0$.

Examples: Every polynomial function p is differentiable on $\mathbb{R}$. A rational function p/q is differentiable at c if $q(c) \neq 0$.

Differentiability: Chain Rule

Theorem

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. Also, let $E \subset \mathbb{R}$ and $g : E \to \mathbb{R}$ be such that $f(D) \subseteq E$. Suppose c is an interior point of D and $f(c)$ is an interior point of E. If f is differentiable at c, and if g is differentiable at $f(c)$, then $g \circ f : D \to \mathbb{R}$ is differentiable at c, and $(g \circ f)'(c) = g'(f(c))f'(c)$.

Proof. Let f_1 be the increment functions associated with f and c, and let g_1 be the increment functions associated with g and $d := f(c)$. Then $f(x) - f(c) = (x - c)f_1(x)$ for all $x \in D$, where f_1 is continuous at c, and $g(y) - g(d) = (y - d)g_1(y)$ for all $y \in E$, where g_1 is continuous at d. Now let us consider $h := g \circ f : D \to \mathbb{R}$. Then for $x \in D$,

$$\begin{aligned} h(x) - h(c) &= g(f(x)) - g(f(c)) \\ &= \big(f(x) - f(c)\big)g_1(f(x)) = (x - c)f_1(x)(g_1 \circ f)(x). \end{aligned}$$

Let $h_1 := f_1 \cdot (g_1 \circ f) : D \to \mathbb{R}$. Then h_1 is continuous at c since f and f_1 are continuous at c and g_1 is continuous at $d = f(c)$. It follows that h_1 is the increment function associated with $g \circ f$ and c, and

$$(g \circ f)'(c) = h_1(c) = g'(f(c))f'(c). \qquad \square$$

Conclusion of the Chain Rule in the Leibnitz notation:

If $y := f(x)$ and $z := g(y)$, then $z = (g \circ f)(x)$ and

$$\left.\frac{dz}{dx}\right|_{x=c} = \left.\frac{dz}{dy}\right|_{y=f(c)} \cdot \left.\frac{dy}{dx}\right|_{x=c}$$

Warning: Cancelling out dy in the RHS above to obtain the LHS does not yield a proof of the Chain Rule!

Example: Consider

Let

$$f(x) := \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if} \quad x \in \mathbb{R} \setminus \{0\}, \\ 0 & \text{if} \quad x := 0. \end{cases}$$

$$f_1(x) := \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if} \quad x \in \mathbb{R} \setminus \{0\}, \\ 0 & \text{if} \quad x := 0. \end{cases}$$

Then f_1 is continuous at 0, and $f(x) - f(0) = (x - 0)f_1(x)$ for all $x \in \mathbb{R}$. By the C-lemma, f is differentiable at 0, and $f'(0) = f_1(0) = 0$. Also, if $c \neq 0$, then f is differentiable at c, and by product rule and chain rule, $f'(c) = 2c \sin \frac{1}{c} - \cos \frac{1}{c}$.

Note: The derivative f' of f is not continuous at 0. Why?

Differentiability of the inverse function

Theorem

Let I be an interval, and let c be an interior point of I. If $f : I \to \mathbb{R}$ is a one-one continuous function such that f is differentiable at c and $f'(c) \neq 0$, then $f^{-1} : f(I) \to \mathbb{R}$ is differentiable at $f(c)$ and

$$\left(f^{-1}\right)'\left(f(c)\right) = \frac{1}{f'(c)}.$$

Proof: Skipped. You may try, if interested. Or look up [GL-1].

Note: Since $f : I \to \mathbb{R}$ is continuous, it has the IVP, and so $J = f(I)$ is an interval. Hence $f(c)$ is either an interior point of J or an end-point of J. Since f is also one-one, it can be shown to be strictly monotonic (to be defined later), and so $f(c)$ is in fact an interior point of J.