MA 105 : Calculus
Division 3, Lecture 07

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Recap of the previous lecture

- Right hand and left hand derivatives
- Higher order derivatives
- Equivalent condition for Differentiability: C-Lemma
- Consequences of the Carathéodory Lemma (C-Lemma):
  - Differentiability implies Continuity
  - Algebraic Rules (concerning sums, products, and quotients of differentiable functions)
  - Chain Rule.
  - Differentiability of the inverse function

Differentiability of the inverse function

Theorem

Let I be an interval, and let c be an interior point of I. If $f : I \to \mathbb{R}$ is a one-one continuous function such that f is differentiable at c and $f'(c) \neq 0$, then $f^{-1} : f(I) \to \mathbb{R}$ is differentiable at $f(c)$ and

$$\left(f^{-1}\right)'\left(f(c)\right) = \frac{1}{f'(c)}.$$

Proof: Skipped. You may try, if interested. Or look up [GL-1].

Note: Since $f : I \to \mathbb{R}$ is continuous, it has the IVP, and so $J = f(I)$ is an interval. Hence $f(c)$ is either an interior point of J or an end-point of J. Since f is also one-one, it can be shown to be strictly monotonic (to be defined later), and so $f(c)$ is in fact an interior point of J.

Examples of derivatives of inverse functions

(i) Let $n \in \mathbb{N}$, and let $f(x) := x^n$ for $x \in (0, \infty)$. Then f is one-one and continuous. Consider $c \in (0, \infty)$. Now $f'(c) = n\,c^{n-1} \neq 0$. Further, $f\big((0, \infty)\big) = (0, \infty)$. If $d \in (0, \infty)$ and $f(c) = c^n = d$, then $c = d^{1/n}$ and

$$\big(f^{-1}\big)'(d) = \frac{1}{f'(c)} = \frac{1}{n\,c^{n-1}} = \frac{1}{n\,d^{(n-1)/n}} = \frac{1}{n} d^{(1/n)-1}.$$

(ii) Let $f(x) := \sin x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then f is one-one and continuous. Consider $c \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Now $f'(c) = \cos c \neq 0$. Further, $f\big((-\frac{\pi}{2}, \frac{\pi}{2})\big) = (-1, 1)$. If $d \in (-1, 1)$ and $f(c) = \sin c = d$, then

$$\big(f^{-1}\big)'(d) = \frac{1}{f'(c)} = \frac{1}{\cos c} = \frac{1}{\sqrt{1 - \sin^2 c}} = \frac{1}{\sqrt{1 - d^2}}.$$

Local Extremum

Let $D \subseteq \mathbb{R}$, and let c be an interior point of D.
We say that a function $f : D \to \mathbb{R}$ has

(i) a **local minimum** at c if there is $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq D$ and

$$f(x) \geq f(c) \quad \text{for all } x \in (c - \delta, c + \delta).$$

(ii) a **local maximum** at c if there is $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq D$ and

$$f(x) \leq f(c) \quad \text{for all } x \in (c - \delta, c + \delta).$$

(iii) a **local extremum** at c if it has a local maximum or a local minimum at c.

Proposition

Let $D \subseteq \mathbb{R}$, and let c be an interior point of D. If $f : D \to \mathbb{R}$ has a local extremum at c, and if f is differentiable at c, then $f'(c) = 0$.

Proof.

Let f be differentiable at c. By the C-lemma, there is $f_1 : D \to \mathbb{R}$ such that f_1 is continuous at c and $f(x) - f(c) = (x - c)f_1(x)$ for all $x \in D$.

Let f have a local minimum at c. Then there is $\delta > 0$ such that $f(x) \geq f(c)$ for all $x \in (c - \delta, c + \delta)$. Hence $f_1(x) \geq 0$ for all $x \in (c, c + \delta)$ and $f_1(x) \leq 0$ for all $x \in (c - \delta, c)$.

Now $0 \leq \lim_{x \to c^+} f_1(x) = f_1(c) = \lim_{x \to c^-} f_1(x) \leq 0$, since f_1 is continuous at c. Hence $f'(c) = f_1(c) = 0$.

A similar argument holds if f has a local maximum at c. □

Examples:

(i) Let $D := [0,1]$, and $f(x) := x$ for $x \in D$. Then $f(0) = \min\{f(x) : x \in D\}$, $f(1) = \max\{f(x) : x \in D\}$. But $f'_+(0) = 1 \neq 0$ and $f'_-(1) = 1 \neq 0$. Note: Neither 0 nor 1 is an interior point of D.

(ii) Let $D := [-1,1]$, and $f(x) = |x|$ for $x \in D$. Then f has a local minimum at the interior point 0 of D. But $f'_+(0) = 1 \neq 0$ and $f'_-(1) = -1 \neq 0$. Note: f is not differentiable at 0.

(iii) Let $D := [-1,1]$, and $f(x) := x^2$ for $x \in D$. Then f has a local minimum at the interior point 0 of D, and $f'(0) = 0$.

(iv) Let $D := [-1,1]$, and $f(x) := x^3$ for $x \in D$. Then 0 is an interior point of D and $f'(0) = 0$. But f does not have a local extremum at 0. Thus the converse of the above proposition is false.

Rolle's Theorem

Theorem

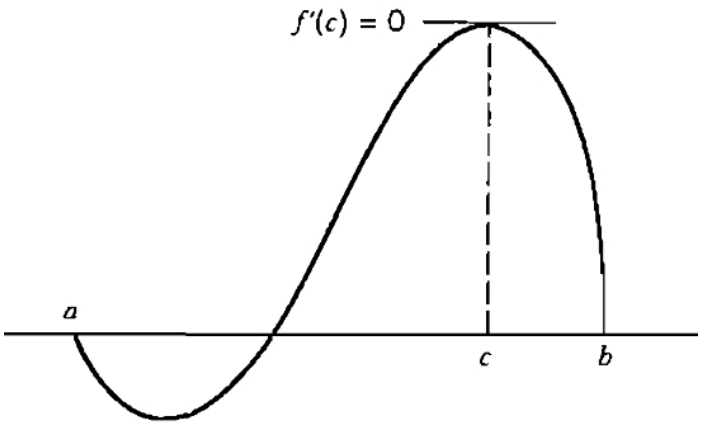
Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a function such that

(i) f is continuous on $[a, b]$,

(ii) f is differentiable on (a, b), and

(iii) $f(a) = f(b)$.

Then there exists $c \in (a, b)$ such that $f'(c) = 0$.



Thus there exists a point $c \in (a, b)$ such that the tangent to the graph at $(c, f(c))$ is parallel to the x-axis.

Proof. Since f is continuous on $[a, b]$, it is bounded and attains its bounds on $[a, b]$. Let $c_1, c_2 \in [a, b]$ be such that

$$f(c_1) = \min\{f(x) : x \in [a, b]\} \text{ and } f(c_2) = \max\{f(x) : x \in [a, b]\}.$$

If c_1 or c_2 is an interior point of $[a, b]$, then $f'(c_1) = 0$ or $f'(c_2) = 0$ by the previous proposition, and we are done.

If both c_1 and c_2 are end-points of $[a, b]$, then $f(c_1) = f(a) = f(b) = f(c_2)$. Since the minimum and the maximum values of f on $[a, b]$ are the same, f is a constant function on $[a, b]$, and so $f'(c) = 0$ for every $c \in (a, b)$. □

Example:
Let $f(x) := |x|$ for $x \in [-1, 1]$. Then f is continuous on $[-1, 1]$ and $f(-1) = 1 = f(1)$. But there is no $c \in (-1, 1)$ such that $f'(c) = 0$. Note: f is not differentiable at 0.

Application of the Rolle theorem

Let $f(x) := x^4 + 2x^3 - 2$ for $x \in \mathbb{R}$. We show that f has a unique root in $(0, \infty)$, and it is in $(0, 1)$.

- Using the IVP, we have seen that f has a root in (0,1).
- Suppose there are two roots in $(0, \infty)$. Then there are $a, b \in \mathbb{R}$ with $0 < a < b$ such that $f(a) = 0 = f(b)$.
- By Rolle's theorem, there is $c \in (a, b) \subseteq (0, \infty)$ such that $f'(c) = 0$.
- But $f'(x) = 4x^3 + 6x^2 = 2x^2(2x + 3) \neq 0$ for $x \in (0, \infty)$.
- Hence f has exactly one (real) root in $(0, \infty)$.

Lagrange's Mean Value Theorem

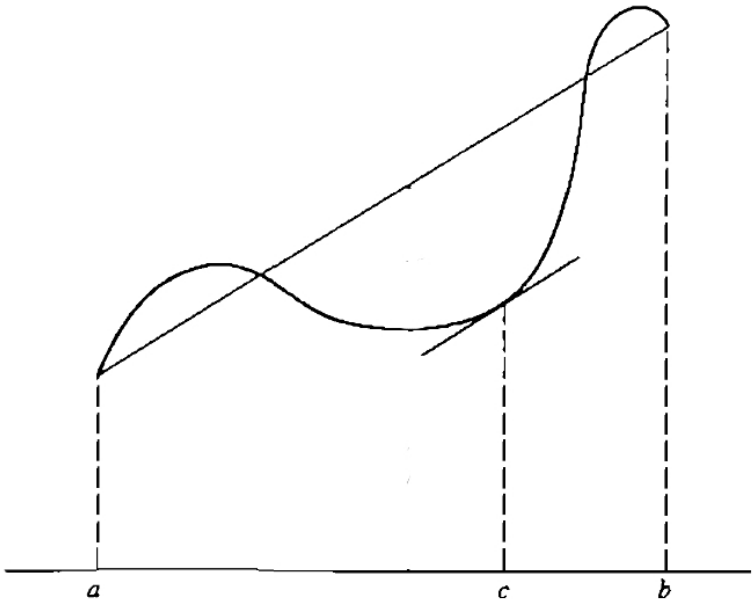
Theorem (MVT)

Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a function such that

(i) f is continuous on $[a, b]$, and

(ii) f is differentiable on (a, b).

Then there exists $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



Thus there exists $c \in (a, b)$ such that the tangent to the graph at $(c, f(c))$ is parallel to the line joining $(a, f(a))$ and $(b, f(b))$.

Proof. For $x \in [a, b]$, define

$$F(x) := f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then F is continuous on $[a, b]$, differentiable on (a, b) and $F(a) = f(a) = F(b)$. By Rolle's theorem, there is $c \in (a, b)$ such that $F'(c) = 0$, that is, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. □

Note: We have used the Rolle theorem to prove a more general result, namely the Lagrange mean value theorem.

Physical Interpretation of MVT: For $t \in [a, b]$, let $f(t)$ denote the distance travelled by a particle at time t. Then the average speed of the particle is $\dfrac{f(b) - f(a)}{b - a}$, and its instantaneous speed at time $t \in [a, b]$ is $f'(t)$. Thus there is a time $c \in [a, b]$ at which the average speed equals the instantaneous speed.

Consequences of the Mean Value Theorem (MVT)

Proposition

Let I be an interval and $f : I \to \mathbb{R}$. If $f'(x)$ exists and equals 0 for all $x \in I$, then f is constant on I.

Proof. Let x_1, x_2 in I with $x_1 < x_2$. Then $[x_1, x_2] \subseteq I$.
By the MVT,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \quad \text{for some } c \in (x_1, x_2).$$

Since $f'(c) = 0$, we obtain $f(x_2) = f(x_1)$. Thus f has the same value at every point of I. □

We have already seen that the converse is also true.

Example: Let $D := \mathbb{R} \setminus \{0\}$, and $f(x) := x/|x|$ for $x \in D$. Then $f' = 0$ on D, but f is not constant on D; this is because D is not an interval.

Consequences of MVT (continued)

Monotonicity

Let I be an interval, and let $f : I \to \mathbb{R}$. We say that f is

- **(monotonically) increasing** on I if
  $x_1, x_2 \in I,\ x_1 < x_2 \implies f(x_1) \leq f(x_2)$.
- **(monotonically) decreasing** on I if
  $x_1, x_2 \in I,\ x_1 < x_2 \implies f(x_1) \geq f(x_2)$.
- **monotonic** on I if it is increasing on I, or it is decreasing on I.
- **strictly increasing** on I if
  $x_1, x_2 \in I,\ x_1 < x_2 \implies f(x_1) < f(x_2)$.
- **strictly decreasing** on I if
  $x_1, x_2 \in I,\ x_1 < x_2 \implies f(x_1) > f(x_2)$.
- **strictly monotonic** on I if either it is strictly increasing on I, or it is strictly decreasing on I.

Derivatives and Monotonicity

Proposition

Let I be an interval, and let $f : I \to \mathbb{R}$ be differentiable. Then

(i) $f' \geq 0$ on $I \iff f$ is increasing on I.

(ii) $f' \leq 0$ on $I \iff f$ is decreasing on I.

(iii) $f' > 0$ on $I \implies f$ is strictly increasing on I.

(iv) $f' < 0$ on $I \implies f$ is strictly decreasing on I.

All forward ($\implies$) implications follow from the MVT.

The backward ($\impliedby$) implication in (i) follows since $\big(f(x+h) - f(x)\big)/h \geq 0$ for all $h \neq 0$ with $x + h \in I$.
Similarly, the backward ($\impliedby$) implication in (ii) follows.

The converse of (iii) is false: Let $f(x) := x^3$ for $x \in \mathbb{R}$. Then f is strictly increasing on $\mathbb{R}$, but $f'(0) = 0$.
Similarly, the converse of (iv) is false.