MA 105 : Calculus
Division 3, Lecture 08

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Recap of the previous lecture

- Differentiability of the inverse function. Examples
- Local extremum. Basic properties.
- Rolle's theorem. Examples
- Lagrange's Mean Value Theorem (MVT)
- Monotonicity. Relation with derivatives

Monotonicity

Let I be an interval, and let $f : I \to \mathbb{R}$. We say that f is

- **(monotonically) increasing** on I if
  $x_1, x_2 \in I,\ x_1 < x_2 \implies f(x_1) \leq f(x_2)$.
- **(monotonically) decreasing** on I if
  $x_1, x_2 \in I,\ x_1 < x_2 \implies f(x_1) \geq f(x_2)$.
- **monotonic** on I if it is increasing on I, or it is decreasing on I.
- **strictly increasing** on I if
  $x_1, x_2 \in I,\ x_1 < x_2 \implies f(x_1) < f(x_2)$.
- **strictly decreasing** on I if
  $x_1, x_2 \in I,\ x_1 < x_2 \implies f(x_1) > f(x_2)$.
- **strictly monotonic** on I if either it is strictly increasing on I, or it is strictly decreasing on I.

Derivatives and Monotonicity

Proposition

Let I be an interval, and let $f : I \to \mathbb{R}$ be differentiable. Then

(i) $f' \geq 0$ on $I \iff f$ is increasing on I.

(ii) $f' \leq 0$ on $I \iff f$ is decreasing on I.

(iii) $f' > 0$ on $I \implies f$ is strictly increasing on I.

(iv) $f' < 0$ on $I \implies f$ is strictly decreasing on I.

All forward ($\implies$) implications follow from the MVT.

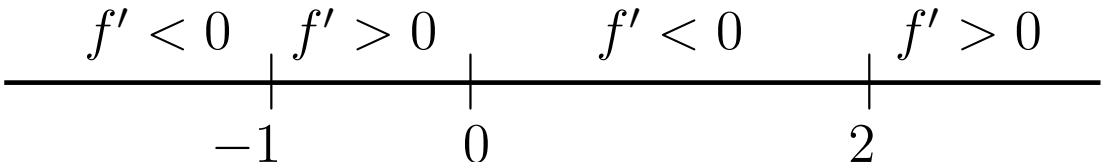
The backward ($\impliedby$) implication in (i) follows since $\big(f(x+h) - f(x)\big)/h \geq 0$ for all $h \neq 0$ with $x + h \in I$.
Similarly, the backward ($\impliedby$) implication in (ii) follows.

The converse of (iii) is false: Let $f(x) := x^3$ for $x \in \mathbb{R}$. Then f is strictly increasing on $\mathbb{R}$, but $f'(0) = 0$.
Similarly, the converse of (iv) is false.

Examples

(i) Let $f(x) := 3x^4 - 4x^3 - 12x^2 + 5$ for $x \in \mathbb{R}$. Let us find intervals on which f is (strictly) increasing, or (strictly) decreasing. For $x \in \mathbb{R}$,

$$\begin{aligned} f'(x) &= 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) \\ &= 12(x+1)x(x-2). \end{aligned}$$



- f is strictly decreasing on $(-\infty, -1)$.
- f is strictly increasing on $(-1, 0)$.
- f is strictly decreasing on $(0, 2)$.
- f is strictly increasing on $(2, \infty)$.

(ii) Let $f(x) := x - \sin x$ for $x \in [0, \pi/2]$. Then $f'(x) = 1 - \cos x \geq 0$ for all $x \in [0, \pi/2]$. Hence f is increasing on $[0, \pi/2]$, and so $0 = f(0) \leq f(x)$ for $x \in [0, \pi/2]$. Thus $x \in [0, \pi/2] \Longrightarrow \sin x \leq x$. In fact, $\sin x < x$ for $x \in (0, \pi/2]$ since $f'(x) > 0$ for $x \in (0, \pi/2]$.

(iii) Let $g(x) := x - 1 + \cos x$ for $x \in [0, \pi/2]$. Then $g'(x) = 1 - \sin x \geq 0$ for all $x \in [0, \pi/2]$. Hence g is increasing on $[0, \pi/2]$, and so $0 = g(0) \leq g(x)$ for $x \in [0, \pi/2]$. Thus $x \in [0, \pi/2] \Longrightarrow 1 - \cos x \leq x$. In fact, since $g'(x) > 0$ for $x \in [0, \pi/2)$, we see that $g(0) < g(x)$ for $x \in (0, \pi/2)$, that is, $1 - \cos x < x$ for $x \in (0, \pi/2)$.

(iv) Let $h(x) := \sin x - x \cos x$ for $x \in [0, \pi/2]$. Then $h'(x) = x \sin x \geq 0$ for all $x \in [0, \pi/2]$. Hence h is increasing on $[0, \pi/2]$, and so $0 = h(0) \leq h(x)$ for $x \in [0, \pi/2]$. Thus $x \in [0, \pi/2] \Longrightarrow x \cos x \leq \sin x$. In fact, $x \cos x < \sin x$ for $x \in (0, \pi/2]$ since $h'(x) > 0$ for $x \in (0, \pi/2]$.

The above 3 results yield the inequalities quoted in Lecture 4.

Convexity and Concavity

Let I be an interval. A function $f : I \to \mathbb{R}$ is called **convex** on I if the line segment joining any two points on the graph of f lies on or above the graph of f. If $x_1, x_2 \in I$, then the equation of the line passing through $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by $y = f(x_1) + \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$. Thus f is **convex** on I if

$$x_1, x_2,\ x \in I,\ x_1 < x < x_2 \Longrightarrow f(x) \leq f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1).$$

Similarly, a function $f : I \to \mathbb{R}$ is called **concave** on I if

$$x_1, x_2,\ x \in I,\ x_1 < x < x_2 \Longrightarrow f(x) \geq f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1).$$

Remark: It is easy to see that f

$$f\left((1-\lambda)x_1 + \lambda x_2\right) \leq (1-\lambda)f(x_1) + \lambda f(x_2)\ \forall\, \lambda \in [0,1] \text{ and } x_1, x_2 \in I.$$

Likewise for concacity (with $\leq$ changed to $\geq$).

A function $f : I \to \mathbb{R}$ is called **strictly convex** on I if the inequality $\leq$ in the definition of a convex function can be replaced by the strict inequality $<$.

A function $f : I \to \mathbb{R}$ is called **strictly concave** on I if the inequality $\geq$ in the definition of a concave function can be replaced by the strict inequality $>$.

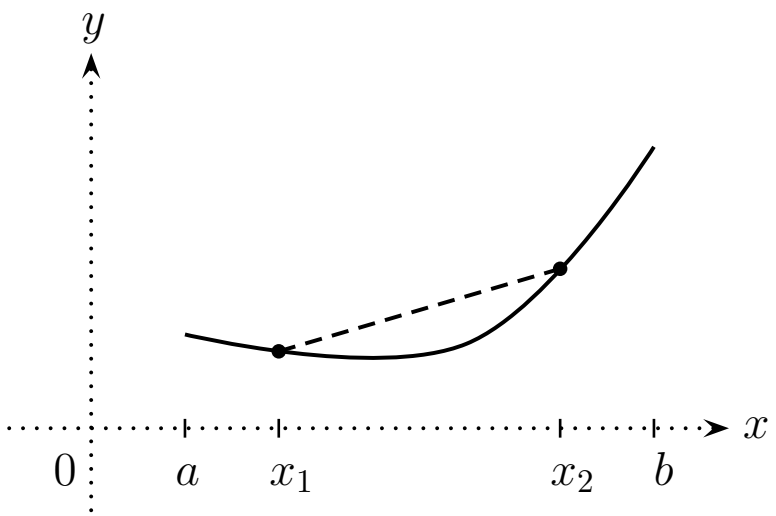
Examples:

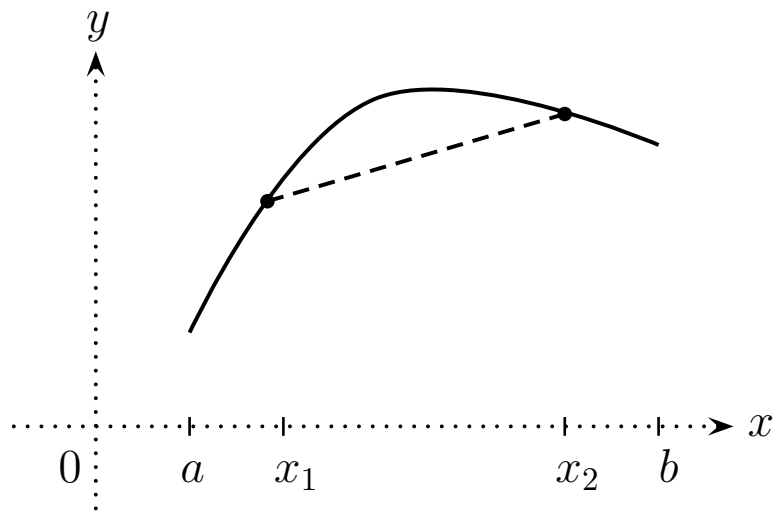
(i) Let $\alpha, \beta \in \mathbb{R}$, and let $f(x) := \alpha + \beta x$ for $x \in \mathbb{R}$. Then f is convex as well as concave on $\mathbb{R}$ (but not strictly).

(ii) Let $\alpha \in \mathbb{R}$, and let $f(x) := \alpha |x|$ for $x \in \mathbb{R}$. Then f is convex on $\mathbb{R}$ if $\alpha > 0$, and f is concave on $\mathbb{R}$ if $\alpha < 0$ (but not strictly). (Prove this from the first principles.)

(iii) Let $\alpha \in \mathbb{R}$, and let $f(x) := \alpha x^2$ for $x \in \mathbb{R}$. Then f is strictly convex on $\mathbb{R}$ if $\alpha > 0$, and f is strictly concave on $\mathbb{R}$ if $\alpha < 0$. (Prove this from the first principles.)

Typical graphs of convex and concave functions on $I := [a, b]$ look as follows:





Let I be an interval, and let $f : I \to \mathbb{R}$ be a function. Suppose the curve $y = f(x)$ has a (unique) nonvertical tangent at each point on it.

It is apparent that if f is convex, then the slopes of the tangents increase as we move from left to right, and if f is concave, then then the slopes of the tangents decrease as we move from left to right. This is made precise in the following.

Proposition

Let I be an interval and $f : I \to \mathbb{R}$ be differentiable. Then

(i) f' is increasing on $I \iff f$ is convex on I.

(ii) f' is decreasing on $I \iff f$ is concave on I.

(iii) f' is strictly increasing on $I \iff f$ is strictly convex on I.

(iv) f' is strictly decreasing on $I \iff f$ is strictly concave on I.

We omit proofs of the above results as they are a bit involved.

Corollary

Let I be an interval and $f : I \to \mathbb{R}$ be twice differentiable.

(i) $f'' \geq 0$ on $I \iff f$ is convex on I.

(ii) $f'' \leq 0$ on $I \iff f$ is concave on I.

(iii) $f'' > 0$ on $I \implies f$ is strictly convex on I.

(iv) $f'' < 0$ on $I \implies f$ is strictly concave on I.

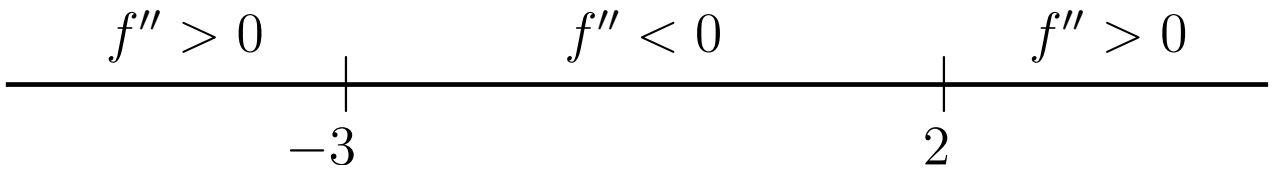
The converse of (iii) in the above corollary is false. To see this, consider $f(x) := x^4$ for $x \in \mathbb{R}$. Then f is strictly convex on $\mathbb{R}$, but $f''(0) = 0$. Similarly, the converse of (iv) is false.

Example:

Let $f(x) := x^4 + 2x^3 - 36x^2 + 62x + 5$ for $x \in \mathbb{R}$. Let us find intervals on which f is (strictly) convex or (strictly) concave. For $x \in \mathbb{R}$,

$$\begin{aligned} f'(x) &= 4x^3 + 6x^2 - 72x + 62, \text{ and} \\ f''(x) &= 12x^2 + 12x - 72 = 12(x+3)(x-2). \end{aligned}$$

Note:



- f is strictly convex on $(-\infty, -3)$.
- f is strictly concave on $(-3, 2)$.
- f is strictly convex on $(2, \infty)$.

Critical Points and Global Extrema

Let $D \subset \mathbb{R}$, and let $f : D \to \mathbb{R}$. An interior point c of D is called a **critical point** of f if either f is not differentiable at c, or if f is differentiable at c and $f'(c) = 0$.

Proposition

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then the global minimum $m := \min\{f(x) : x \in [a, b]\}$ as well as the global maximum $M := \max\{f(x) : x \in [a, b]\}$ of f on $[a, b]$ is attained either at a critical point of f or at an end-point of $[a, b]$.

Proof. There is $c_1 \in [a, b]$ such that $f(c_1) = m$. If $c_1 = a$ or $c_1 = b$, then we are done. Next, let $c_1 \in (a, b)$. Then f has a local minimum at c_1. If f is not differentiable at c_1, then c_1 is a critical point of f. If f is differentiable at c_1, then $f'(c_1) = 0$ by an earlier result, and so c_1 is a critical point of f.
Similarly, we argue for the global maximum M of f. □

Finding Global Extrema

Example: Consider $f : [-1, 2] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} -x & \text{if } -1 \le x \le 0, \\ 2x^3 - 4x^2 + 2x & \text{if } 0 < x \le 2. \end{cases}$$

f is continuous on $[-1, 2]$: $\lim_{x \to 0^-} f(x) = f(0) = \lim_{x \to 0^+} f(x)$.
f is not differentiable at 0 : $f'_-(0) = -1$ and $f'_+(0) = 2$.
For $-1 < x < 0$, $f'(x) = -1 \neq 0$.
For $0 < x < 2$, $f'(x) = 6x^2 - 8x + 2 = 2(3x - 1)(x - 1)$.
Critical points of f: 0, $1/3$, 1. End-points of $[-1, 2]$: -1, 2.

x	-1	0	$1/3$	1	2
$f(x)$	1	0	$8/27$	0	4

f attains its global maximum 4 at $x = 2$, and its global minimum 0 at $x = 0$ as well as at $x = 1$.

Moral of the story

While finding global extrema of a continuous function f defined on $[a, b]$, it is of no use to know whether a local extremum of f is in fact a local maximum or a local minimum. Hence do not find the second derivative of f.

To determine the global extrema of f, find the critical points of f, that is, the points at which the derivative of f does not exist and the points in (a, b) at which the derivative of f exists and is equal to 0. Then find the values of f at the critical points of f, and at the end points a and b of $[a, b]$.

Compare these values of f. The smallest among them is the global minimum of f and the largest among them is the global maximum of f.

Local Extrema and Derivatives

Intuitively, it is clear that if the slopes of tangents to a curve $y = f(x)$ are positive to the left of $x = c$ and negative to the right of $x = c$, then f has a local maximum at c. A more precise and general formulation of this is the following.

Theorem (First derivative test for a local maximum)

Let $D \subset \mathbb{R}$, c an interior point of D, and $f : D \to \mathbb{R}$. Suppose

(i) f is continuous at c, and

(ii) there is $\delta > 0$ such that $f' \geq 0$ on $(c - \delta, c)$ and $f' \leq 0$ on $(c, c + \delta)$.

Then f has a local maximum at c.

Proof: By (ii), f is increasing on $(c - \delta, c)$, and it is decreasing on $(c, c + \delta)$. Also, $\lim_{h \to 0} f(c + h) = f(c)$ by (i). Hence $x \in (c - \delta, c) \implies f(x) \leq \lim_{h \to 0^-} f(c + h) = f(c)$ and $x \in (c, c + \delta) \implies f(x) \leq \lim_{h \to 0^+} f(c + h) = f(c)$. Hence f has a local maximum at c. □

Theorem (First derivative test for a local minimum)

Let $D \subset \mathbb{R}$, c an interior point of D, and $f : D \to \mathbb{R}$. Suppose

(i) f is continuous at c, and

(ii) there is $\delta > 0$ such that $f' \leq 0$ on $(c - \delta, c)$ and $f' \geq 0$ on $(c, c + \delta)$.

Then f has a local minimum at c.

Proof: Similar to the proof of the previous result.

Examples:

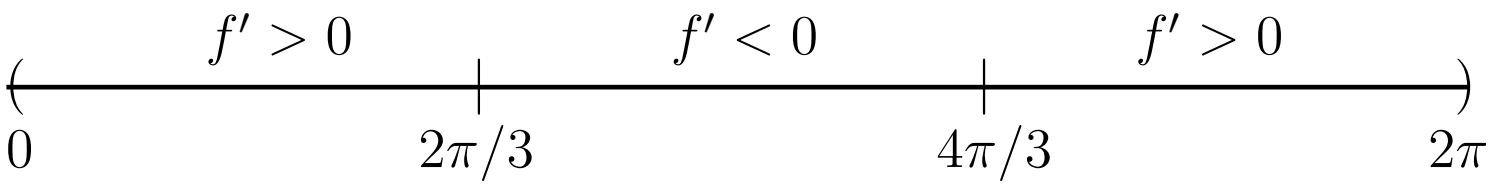
(i) Let $f(x) := |x|$ for $x \in \mathbb{R}$. Then f is continuous at 0, $f' = -1 \leq 0$ on $(-\infty, 0)$ and $f' = 1 \geq 0$ on $(0, \infty)$. Thus f has a local minimum at 0.

(ii) Let $f(x) := -x - 1$ for $x \in (-\infty, 0)$, $f(0) := 0$ and $f(x) := x + 1$ for $x \in (0, \infty)$. Then $f' = -1 \leq 0$ on $(-\infty, 0)$ and $f' = 1 \geq 0$ on $(0, \infty)$. But f does not have a local minimum at 0. Note: f is not continuous at 0.

(iii) Let $f(x) := x + 2\sin x$ for $x \in (0, 2\pi)$.

Then $f'(x) = 1 + 2\cos x = 0 \iff x \in \{2\pi/3, 4\pi/3\}$.

f is continuous at $2\pi/3$ and at $4\pi/3$. We note the following:



f has a local maximum at $2\pi/3$ and a local minimum at $4\pi/3$.

In general, the first derivative test for a local extremum can be informally summarized by the following Thumb Rule.

Suppose f is continuous at c.

f' changes from $+$ to $-$ at $c \implies$ f has a local maximum at c,

f' changes from $-$ to $+$ at $c \implies$ f has a local minimum at c.