MA 105 : Calculus
Division 3, Lecture 09

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Recap of the previous lecture

- Monotonicity. Relation with derivatives
- Convexity and Concavity. Relation with derivatives
- Critical Points and Global Extrema
- Local Extrema and Derivatives

Local Extrema and Derivatives

Intuitively, it is clear that if the slopes of tangents to a curve $y = f(x)$ are positive to the left of $x = c$ and negative to the right of $x = c$, then f has a local maximum at c. A more precise and general formulation of this is the following.

Theorem (First derivative test for a local maximum)

Let $D \subset \mathbb{R}$, c an interior point of D, and $f : D \to \mathbb{R}$. Suppose

(i) f is continuous at c, and

(ii) there is $\delta > 0$ such that $f' \geq 0$ on $(c - \delta, c)$ and $f' \leq 0$ on $(c, c + \delta)$.

Then f has a local maximum at c.

Proof: By (ii), f is increasing on $(c - \delta, c)$, and it is decreasing on $(c, c + \delta)$. Also, $\lim_{h \to 0} f(c + h) = f(c)$ by (i). Hence $x \in (c - \delta, c) \implies f(x) \leq \lim_{h \to 0^-} f(c + h) = f(c)$ and $x \in (c, c + \delta) \implies f(x) \leq \lim_{h \to 0^+} f(c + h) = f(c)$. Hence f has a local maximum at c. □

Theorem (First derivative test for a local minimum)

Let $D \subset \mathbb{R}$, c an interior point of D, and $f : D \to \mathbb{R}$. Suppose

(i) f is continuous at c, and

(ii) there is $\delta > 0$ such that $f' \leq 0$ on $(c - \delta, c)$ and $f' \geq 0$ on $(c, c + \delta)$.

Then f has a local minimum at c.

Proof: Similar to the proof of the previous result.

Examples:

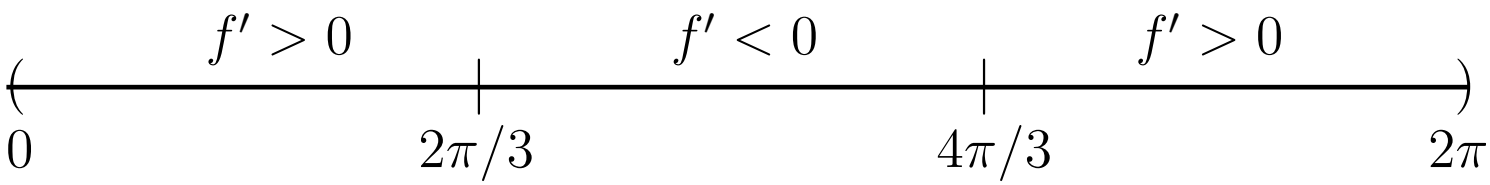
(i) Let $f(x) := |x|$ for $x \in \mathbb{R}$. Then f is continuous at 0, $f' = -1 \leq 0$ on $(-\infty, 0)$ and $f' = 1 \geq 0$ on $(0, \infty)$. Thus f has a local minimum at 0.

(ii) Let $f(x) := -x - 1$ for $x \in (-\infty, 0)$, $f(0) := 0$ and $f(x) := x + 1$ for $x \in (0, \infty)$. Then $f' = -1 \leq 0$ on $(-\infty, 0)$ and $f' = 1 \geq 0$ on $(0, \infty)$. But f does not have a local minimum at 0. Note: f is not continuous at 0.

(iii) Let $f(x) := x + 2\sin x$ for $x \in (0, 2\pi)$.

Then $f'(x) = 1 + 2\cos x = 0 \iff x \in \{2\pi/3, 4\pi/3\}$.

f is continuous at $2\pi/3$ and at $4\pi/3$. We note the following:



f has a local maximum at $2\pi/3$ and a local minimum at $4\pi/3$.

In general, the first derivative test for a local extremum can be informally summarized by the following Thumb Rule.

Suppose f is continuous at c.

f' changes from $+$ to $-$ at $c \implies$ f has a local maximum at c,

f' changes from $-$ to $+$ at $c \implies$ f has a local minimum at c.

Theorem (Second derivative test for a local extremum)

Let $D \subset \mathbb{R}$, and let c be an interior point of D. Suppose $f : D \to \mathbb{R}$ is twice differentiable at c, and $f'(c) = 0$.

(i) If $f''(c) < 0$, then f has a local maximum at c.

(ii) If $f''(c) > 0$, then f has a local minimum at c.

Proof: (i) Let $f''(c) < 0$. Since $f''(c)$ exists, there is $r > 0$ such that f' exits on $(c - r, c + r)$. Now

$$\lim_{x \to c} \frac{f'(x)}{x - c} = \lim_{x \to c} \frac{f'(x) - f'(c)}{x - c} = f''(c) < 0.$$

Hence there is $\delta > 0$ such that $f'(x)/(x - c) < 0$ for all $x \in (c - \delta, c) \cup (c, c + \delta)$. Now $x \in (c - \delta, c) \implies f'(x) > 0$ and $x \in (c, c + \delta) \implies f'(x) < 0$. By the first derivative test, f has a local maximum at c. A similar argument works for (ii). □

Examples:

(i) Let $f(x) := x^4 - 2x^2$ for $x \in \mathbb{R}$. Then

$f'(x) = 4x^3 - 4x = 4(x+1)x(x-1) = 0 \iff x \in \{-1, 0, 1\}$.

f is twice differentiable, and $f''(x) = 4(3x^2 - 1)$ for $x \in \mathbb{R}$.

Since $f''(-1) = 8 > 0$, $f''(0) = -4 < 0$ and $f''(1) = 8 > 0$,
f has a local maximum at 0, and has a local minimum at ± 1.

(ii) Let $f(x) := x^4$ for $x \in (-1, 1)$.
Then $f'(x) = 4x^3$ and $f''(x) = 12x^2$ for $x \in (-1, 1)$,
and so $f(0) = f'(0) = f''(0) = 0$.

Hence the second derivative test is not applicable at 0. But since $f' < 0$ on $(-1, 0)$ and $f' > 0$ on $(0, 1)$, the first derivative test shows that f has a local minimum at 0.

Note: The first derivative test is more general than the second.

Point of Inflection

Let I be an interval, and let $f : I \to \mathbb{R}$. An interior point of I at which convexity of f changes to concavity, or the other way round, is called a point of inflection for f. More precisely,

Definition

An interior point c of I is called a **point of inflection** for f if there is $\delta > 0$ such that
either f is convex on $(c - \delta, c)$ and concave on $(c, c + \delta)$,
or f is concave on $(c - \delta, c)$ and convex on $(c, c + \delta)$.

Examples:

Let $f(x) := x^3$ and $g(x) := x^{1/3}$ for $x \in \mathbb{R}$. Then 0 is a point of inflection for f as well as for g.

We give characterizations of a point of inflection of a differentiable and of a twice differentiable function.

Point of Inflection and Derivatives

Theorem (Derivative tests for a point of inflection)

Let c be an interior point of I, and let $f : I \to \mathbb{R}$.

(i) (First derivative test) Suppose f is differentiable on $(c-r, c) \cup (c, c+r)$ for some $r > 0$. Then c is point of inflection for $f \iff$ there is $\delta > 0$ with $\delta < r$ such that f' is increasing on $(c-\delta, c)$ and f' is decreasing on $(c, c+\delta)$, or vice-versa.

(ii) (Second derivative test) Suppose f is twice differentiable on $(c-r, c) \cup (c, c+r)$ for some $r > 0$.. Then c is point of inflection for $f \iff$ there is $\delta > 0$ such that $f'' \geq 0$ on $(c-\delta, c)$ and $f'' \leq 0$ on $(c, c+\delta)$, or vice-versa.

Proof: Use the characterizations of convexity and concavity. □

Thumb Rule:

f'' changes sign at $c \iff c$ is a point of inflection for f.

(Necessary condition for a point of inflection)

Let c be an interior point of I, and let $f : I \to \mathbb{R}$. Suppose f is twice differentiable at c. If c is point of inflection for f, then $f''(c) = 0$.

Proof: Since $f''(c)$ exists, there is $r > 0$ such that f is differentiable on $(c - r, c + r)$. Suppose c is point of inflection for f. By the first derivative test for a point of inflection, there is $\delta > 0$ with $\delta < r$ such that (i) f' is increasing on $(c - \delta, c)$ and f' is decreasing on $(c, c + \delta)$, or (ii) vice-versa. Suppose we are in Case (i). Then for all $x, y \in (c - \delta, c)$ with $x < y$ and for all $z, w \in (c, c + \delta)$ with $z < w$, we must have

$$f'(x) \leq f'(y) \quad \text{and} \quad f'(z) \geq f'(w).$$

Since $f''(c)$ exists, f' is continuous at c. Hence taking limit as $y \to c^-$ and as $z \to c^+$, we see that $f'(x) \leq f'(c)$ for all $x \in (c - \delta, c + \delta)$. Thus f' has a local maximum at c. Consequently, $f''(c) = 0$. The proof in Case (ii) is similar. □

The condition '$f''(c) = 0$' is not sufficient to conclude that c is a point of inflection for f. Example: Let $f(x) := x^4$ for $x \in \mathbb{R}$. Then f is twice differentiable at 0 and $f''(0) = 0$. But 0 is not a point of inflection for f.

(Sufficient condition for a point of inflection)

Let c be an interior point of I, and let $f : I \to \mathbb{R}$. Suppose f is thrice differentiable at c. If $f''(c) = 0$ and $f'''(c) \neq 0$, then c is point of inflection for f.

Proof: Without loss of generality, suppose $f'''(c) < 0$. Then

$$\lim_{x \to c} \frac{f''(x)}{x - c} = \lim_{x \to c} \frac{f''(x) - f''(c)}{x - c} = f'''(c) < 0.$$

Hence there is $\delta > 0$ such that $f''(x)/(x - c) < 0$ for all $x \in (c - \delta, c) \cup (c, c + \delta)$. Now $f'' > 0$ on $(c - \delta, c)$, and $f'' < 0$ on $(c, c + \delta)$. By the second derivative test for a point of inflection, c is point of inflection for f. □

The condition '$f'''(c) \neq 0$' is not necessary for c to be a point of inflection of f. Example: Let $f(x) := x^5$ for $x \in \mathbb{R}$. Then 0 is a point of inflection for f, but $f'''(0) = 0$.

To conclude the topic of local extrema and points of inflection, let us consider the following example.

Let $f(x) := x^4 - 4x^3$ for $x \in \mathbb{R}$. Then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0 \iff x \in \{0, 3\},$$

$$f''(x) = 12x^2 - 24x = 12x(x-2) = 0 \iff x \in \{0, 2\}.$$

Also, $f'''(x) = 24(x-1)$ for $x \in \mathbb{R}$.

Since $f''(0) = 0 = f''(2)$ and $f'''(0) = -f'''(2) = -24 \neq 0$, we see that 0 and 2 are points of inflection for f.

Since $f' \leq 0$ on $(-\infty, 3)$ and $f' \neq 0$ on $(-\infty, 0) \cup (0, 3)$, the function f is strictly decreasing on $(-3, 3)$. Hence f does not have a local extremum at 0. Further, since $f'(3) = 0$ and $f''(3) = 36 > 0$, the function f has a local minimum at 3.

Geometric properties of a function

Let I be an interval. We have so far considered the following geometric properties of a function from I to $\mathbb{R}$.

- Boundedness, and the bounds being attained
- Intermediate value property
- Monotonicity
- Convexity/Concavity
- Having global extrema, having local extrema
- Having points of inflection

Further, we have given analytical conditions on the function (such as continuity, nonnegativity of the derivatives, the vanishing of the derivatives etc.) which imply the above-mentioned geometric properties.
Next, we shall now prepare the groundwork to be able to talk about asymptotes.