Indian Institute of Technology Bombay
MA 001 Preparatory Mathematics I

End Semester Examination

Date: November 19, 2004 | Maximum Marks: 80
Time: 9.30 AM - 12.30 PM | Weightage: 40 %

Instructions

1. Attempt all the questions. The answers are to be written in the answer book provided to you.
2. Number the pages of the answer book as well as any supplements you may use.
3. Begin answer to each question on a new page. Write the page number against the corresponding question number on the index page. All the subparts of a question should be answered together.
4. Please make sure that your name and roll number appears on the answer book and on every supplement that you may use.

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1. Determine the principal value of the argument for each of the following complex numbers.
   (i) $\left(5\cos\frac{\pi}{3} + 5i\sin\frac{\pi}{3}\right)^5 \left(3\cos\frac{\pi}{5} + 3i\sin\frac{\pi}{5}\right)^3$, (ii) $\dfrac{(2+i)^2(4-3i)^3}{(4+3i)^4(2-i)^3}$. [6 points]

2. Let n be a positive integer. Define when a complex number is said to be a primitive nth root of unity. If ω is a primitive 12th root of unity, then compute:
   (i) ω^6, (ii) $\omega + \omega^5 + \omega^7 + \omega^{11}$, (iii) $\omega^2 + \omega^3 + \omega^4 + \omega^6 + \omega^8 + \omega^9 + \omega^{10}$.
   [6 points]

3. Prove that for any positive integer n, we have
   $$\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \cdots + \binom{2n}{2n} = 2^{2n-1}.$$ [6 points]

4. Define the discriminant and the δ-invariant of a conic given by
   $$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$
   Determine the discriminant and the δ-invariant of the conics given by the following. Use this to identify the conic (as an ellipse, parabola, hyperbola or a pair of lines).
   (i) $5x^2 + 6xy + 5y^2 - 4x + 4y - 4 = 0$, (ii) $4x^2 - 4xy + y^2 - 10y - 19 = 0$.
   [8 points]

5. Determine the foci and the eccentricity of the following conics.

   (i) $9x^2 + 25y^2 = 225$, (iii) $x^2 - 9y^2 = 1$, (iii) $x = 4\cosh t$, $y = 3\sinh t$.

   [9 points]

6. Show that the parabola $y^2 = 4ax$ has no parallel tangents. [5 points]

7. Consider the conic defined by

$$41x^2 - 24xy + 34y^2 - 90x + 5y + 25 = 0.$$

   Determine an angle θ such that rotation of axes by angle θ would make the xy term in the equation of the conic to vanish. Find the equation of the conic in the new coordinate system obtained by making a rotation by the angle θ. Use this to identify the conic (as an ellipse, parabola, hyperbola or a pair of lines). [8 points] (Hint: $\tan 2\theta = 2\tan\theta/(1 - \tan^2\theta)$.)

8. Determine the projection of $\mathbf{v} = 3\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ onto $\mathbf{w} = 2\mathbf{i} + \mathbf{j}$. Moreover, write $\mathbf{v}$ as a sum of a vector parallel to $\mathbf{w}$ and a vector orthogonal to $\mathbf{w}$. [5 points]

9. Consider the points $A = (1, -1, 0)$, $B = (2, 1, -1)$, $C = (0, 3, 1))$ and $D = (-1, 1, 2$ in the 3-space $\mathbb{R}^3$. Prove that $ABCD$ is a paralleogram, and find the area of this paralleogram. [5 points]

10. Find the volume of the box (parallelopiped) in the 3-space $\mathbb{R}^3$ determined by the vectors $3\mathbf{i} - 7\mathbf{j} - 8\mathbf{k}$, $2\mathbf{i} + 5\mathbf{k}$ and $4\mathbf{i} + \mathbf{j} + 6\mathbf{k}$. [4 points]

11. Find the rank and the nullity of the following matrices.

$$\text{(i)} \begin{bmatrix} 2 & 3 & 4 & 9 & 14 \\ 3 & 4 & 5 & 10 & 15 \\ 4 & 5 & 6 & 11 & 16 \\ 5 & 6 & 7 & 12 & 17 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 1 & 4 & 7 & -3 \\ -2 & 3 & -6 & 1 \\ 0 & 11 & 8 & -5 \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix},$$

   where in (iii) above, a, b, c are distinct real numbers. [12 points]

12. Use Cramer's Rule to solve the following system of equations.

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_2 + x_3 - x_4 &= 1 \\ x_3 + x_4 - x_5 &= 1 \\ x_5 + x_4 - x_3 &= 1 \\ x_4 + x_3 - x_2 &= 1. \end{aligned}$$

   [6 points]