Indian Institute of Technology Bombay

MA 001 Preparatory Mathematics I

Autumn 2004 SRG

Exercise Set 1

1. Consider the complex numbers $z_1 = 3 + 4i$ and $z_2 = 5 - 2i$. Write each of the following in the form $a + bi$ where $a, b \in \mathbb{R}$.

   (i) $z_1 z_2$, (ii) $(3z_1 - z_2)^2$, (iii) z_1/z_2, (iv) $25z_2/z_1$, (v) $\mathrm{Re}(z_1^3)$, (vi) $\mathrm{Re}(z_1)^3$.

2. Represent each of the following complex in the polar form and plot in the Argand plane.

   (i) $1 + i$, (ii) -10, (iii) $\dfrac{i}{3+3i}$, (iv) $\dfrac{2+i}{5-3i}$, (v) $\dfrac{7-5i}{4i}$, (vi) $\left(\dfrac{6+8i}{4-3i}\right)^2$.

3. Compute the absolute values of

   (i) $(4 + 3i)(3 + 4i)$, (ii) $-2i(3 + i)(2 + 4i)(1 + i)$, (iii) $\dfrac{(3 + 4i)(-1 + 2i)}{(-1 - i)(3 - i)}$.

4. If $z = a + ib \in \mathbb{C}$ with $a, b \in \mathbb{R}$, then find the real and imaginary parts of

   (i) z^2, (ii) z^4, (iii) $\dfrac{1}{z}$, (iv) $\dfrac{1}{z^2}$, (v) $\dfrac{z-1}{z+1}$, (vi) $\dfrac{z-i}{z+i}$.

5. Determine:

   (i) $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^3$, (ii) $\left(\dfrac{-1-i\sqrt{3}}{2}\right)^3$, (iii) $\left(\dfrac{1+i\sqrt{3}}{2}\right)^6$, (iv) $\left(\dfrac{1-i\sqrt{3}}{2}\right)^6$.

6. Determine the principal value of the argument for each of the following complex numbers.

   (i) $1 - i$, (ii) -10, (iii) $-5 + 5i$, (iv) $-\pi - \pi i$, (v) $\sqrt{3} + i$, (vi) $\sqrt{3} - i$.

7. If z is a nonzero complex number, then what is the relation between $\arg \dfrac{1}{z}$ and $\arg z$? Also, what is the relation between $\arg \overline{z}$ and $\arg z$? Moreover, what is the relation between $\dfrac{1}{z}$ and $\overline{z}$? Justify your answers.

8. If $z, w \in \mathbb{C}$ are written in polar form as $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\alpha + i\sin\alpha)$, and if $w \neq 0$, then show that the polar form of $\dfrac{z}{w}$ is $\dfrac{r}{s}\left(\cos(\theta - \alpha) + i\sin(\theta - \alpha)\right)$.

9. Compute:

   (i) $\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$, (ii) $\dfrac{12\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)}{4\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)}$.

10. Find all possible values of i^n where $n \in \mathbb{N}$.

11. Let $z = a+bi \in \mathbb{C}$ with $a, b \in \mathbb{R}$. Show that if $x, y \in \mathbb{R}$ satisfy the equation $(x+iy)^2 = z$, then we must have $x^2 + y^2 = \sqrt{a^2+b^2}$. Deduce that

$$x^2 = \frac{1}{2}\left(a + \sqrt{a^2+b^2}\right) \quad \text{and} \quad y^2 = \frac{1}{2}\left(-a + \sqrt{a^2+b^2}\right).$$

Use this to find all possible $w \in \mathbb{C}$ such that $w^2 = z$.

12. Given any $z_1, z_2 \in \mathbb{C}$, prove that

(i) $\big||z_1| - |z_2|\big| \le |z_1 - z_2|$, (ii) $\big||z_1| - |z_2|\big| \le |z_1 + z_2|$, (iii) $\big||z_1| - |z_2|\big| \le |z_1| + |z_2|$.

13. Given any $z_1, z_2 \in \mathbb{C}$, prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$.

14. If $z, w \in \mathbb{C}$ are such that either $|z| = 1$ or $|w| = 1$ (but not both), then show that $\overline{z}w \neq 1$. Further show that

$$\left|\frac{z-w}{1-\overline{z}w}\right| = 1.$$

15. If a complex number z is a root of a polynomial with real coefficients, then prove that its conjugate $\overline{z}$ is also a root.

16. If $z \in \mathbb{C}$ is such that $|z| = 1$, then compute $|1+z|^2 + |1-z|^2$.

17. Describe geometrically the set of points z in $\mathbb{C}$ satisfying the following conditions.

(i) $|z - i| = 2$, (ii) $|z + 3i| \le 1$, (iii) $\mathrm{Re}(z) \ge 0$, (iv) $\mathrm{Im}(z) \ge 0$, (v) $\mathrm{Im}(z^2) = 2$.

18. Given any $n \in \mathbb{N}$ and $z \in \mathbb{C}$ such that $z \neq 1$, prove that

$$1 + z + z^2 + \ldots + z^{n-1} = \frac{1-z^n}{1-z}.$$

19.* Show that unlike in $\mathbb{R}$, there is no order on the set $\mathbb{C}$ of complex numbers. More precisely, show that there does not exist a subset S of $\mathbb{C}$ satisfying the following two properties.

(i) If $z \in \mathbb{C}$, then either $z \in S$ or $-z \in S$ or $z = 0$.

(ii) If $z_1, z_2 \in S$, then $z_1 + z_2 \in S$ and $z_1 z_2 \in S$.

On the other hand, show that there does exist a subset of $\mathbb{C}$ satisfying (i) [but not (ii)] and a subset of $\mathbb{C}$ satisfying (ii) [but not (i)].

20.* Given any $n \in \mathbb{N}$ and complex numbers $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$, prove that

$$\left|\sum_{i=1}^{n} z_i w_i\right|^2 = \left(\sum_{i=1}^{n} |z_i|^2\right)\left(\sum_{i=1}^{n} |w_i|^2\right) - \sum_{1 \le i < j \le n} |z_i \overline{w}_j - z_j \overline{w}_i|^2.$$

Deduce that

$$\left|\sum_{i=1}^{n} z_i w_i\right| \le \left(\sqrt{\sum_{i=1}^{n} |z_i|^2}\right)\left(\sqrt{\sum_{i=1}^{n} |w_i|^2}\right)$$

and moreover, that the equality holds if and only if the z_i are proportional to $\overline{w}_i$ for $i = 1, \ldots, n$, that is, if $z_i \overline{w}_j - z_j \overline{w}_i = 0$ for $i, j = 1, \ldots, n$.