Indian Institute of Technology Bombay
MA 001 Preparatory Mathematics I

Autumn 2004 SRG

Exercise Set 2

1. Find all the cube roots of

   (i) $1+i$, (ii) i, (iii) $1-i$, (iv) $8i$, (v) $-1+i$, (vi) -1.

2. Solve the following equations for Z in $\mathbb{C}$.

   (i) $Z^6 - 2Z^3 - 2iZ^3 + 2i = 0$, (ii) $Z^2 - (5+i)Z + 8 + i = 0$,

   (iii) $Z^4 - (3+6i)Z^2 - 8 + 6i = 0$, (iv) $Z^2 + Z + 1 - i = 0$.

3. Solve the equation $|e^{i\theta} - 1| = 2$ for θ in $[0, 2\pi]$.

4. Write down all the 6th roots of unity in the form $a + ib$. Do the cube roots of unity appear among these? Do the square roots of unity appear among these? Find all possible values of positive integers d for which the dth roots appear among the 6th roots of unity. What can you say about the general case when 6 is replaced by an arbitrary positive integer n?

5. Prove that

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{6\pi}{5} + \cos\frac{8\pi}{5} = -1$$

6. Find all the square roots of

   (i) $i\sqrt{2}$, (ii) $\sqrt{-1+i\sqrt{3}}$, (iii) i^3.

7. Let n be a positive integer. Show that if z is a nonzero complex number, written in polar form as $z = r(\cos\theta + i\sin\theta)$, then

$$z^{-n} = r^{-n}(\cos(-n\theta) + i\sin(-n\theta)).$$

8. Let n be a positive integer. A complex number ω is said to be a primitive nth root of unity if $\omega^n = 1$ and ω 'generates' all the nth roots of unity in the sense that $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are exactly all the nth roots of unity (except possibly for a rearrangement). Prove that

$$\omega_1 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{2\pi i/n}$$

   is always a primitive nth root of unity.

9. Let $n \in \mathbb{N}$ and for $k = 0, 1, \ldots, n-1$, let

$$\omega_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$$

denote, as usual, the nth roots of unity. Determine the values of k, with $0 \le k \le n-1$, for which ω_k is a primitive nth root of unity, when

(i) $n = 2$, (ii) $n = 3$, (iii) $n = 4$, (iv) $n = 5$, (v) $n = 6$,

(vi) $n = 7$, (vii) $n = 8$, (viii) $n = 9$, (ix) $n = 10$, (x) $n = 1$.

Can you guess what would be the answer in the general case?

10. Which powers of a primitive 9th root of unity are the cube roots of unity?

11. Which powers of a primitive 12th root of unity are the cube roots of unity?

12. Let $n \in \mathbb{N}$ and ω be any nth root of unity. Show that the following conditions are equivalent.

    (a) ω is a primitive nth root of unity.

    (b) n is the smallest positive integer for which $\omega^n = 1$, that is, $\omega^d \neq 1$ for $1 \le d < n$ and $\omega^n = 1$.

    (c) The powers $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are distinct.

13. Find the number of primitive roots of unity when

    (i) $n = p$, a prime, (ii) $n = p^2$, where p is a prime.

14. Show that the primitive 18th roots of unity are the negatives of the primitive 9th roots of unity.