Indian Institute of Technology Bombay
MA 001 Preparatory Mathematics I

Autumn 2004 SRG

Exercise Set 3

1. In each of the following, find the quotient $q(x)$ and the remainder $r(x)$ when the polynomial $f(x)$ is divided by the nonzero polynomial $g(x)$.
   (i) $f(x) = 2x^2 + 4x + 3$ and $g(x) = x^2 + 1$,
   (ii) $f(x) = x^2 + 4x + 3$ and $g(x) = 2x^2 + 1$,
   (iii) $f(x) = 3x^4 + 4x^2 + 3x + 5$ and $g(x) = 2x + 3$,
   (iv) $f(x) = 3x^4 + 4x^2 + 3x + 5$ and $g(x) = x^2 + 2$.

2. Find the remainder when the polynomial
$$x^{2004} + x^{2003} + x^{2002} + \ldots + x^2 + x + 1$$
is divided by $x + 1$.

3. If $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ is a polynomial with complex coefficients such that $a_n = -(a_0 + a_1 + \cdots + a_{n-1})$, then prove that 1 is a root of $f(x)$.

4. Show that $\cos((x + 2)/3)$ is not a polynomial in x.

5. If $f(x) \in \mathbb{C}[x]$ is a nonconstant polynomial with complex coefficients and c is any complex number, then show that f takes the value c at only finitely many points, i.e., there are only finitely many $\alpha \in \mathbb{C}$ such that $f(\alpha) = c$. Can you put a bound on the number of $\alpha \in \mathbb{C}$ such that $f(\alpha) = c$?

6. If $f : \mathbb{R} \to \mathbb{R}$ is a periodic function, that is, if there is $p \in \mathbb{R}$ with $p \neq 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$, then show that $f(x)$ can not be a polynomial in x, unless f is a constant function. Does this result hold with $\mathbb{R}$ replaced by $\mathbb{C}$?

7. Show that $\sin x + \cos x$ is not a polynomial in x. Also show that $\sin(x/7) + \cos(x/5)$ is not a polynomial in x.

8. Show that the complex exponential function e^z is not a polynomial in z.

9. A quotient of the form $p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials in $\mathbb{C}[x]$ with $q(x) \neq 0$, is called a rational function in x. Such a rational function in x gives rise to a complex valued function on any subset of $\mathbb{C}$ not containing the roots of $q(x)$. Show that $\sin x$, i.e, the sine function (from $\mathbb{R}$ to $\mathbb{R}$) is not a rational function in x.

10. If $f(x) = ax^2 + bx + c$ is a polynomial in $\mathbb{C}[x]$ of degree ≤ 2, then show that

$$4af(x) = (2ax+b)^2 - (b^2 - 4ac)$$

Use this to derive the formula for the roots of a quadratic, namely, if $a \neq 0$, then the roots of $f(x)$ are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and in particular, $f(x)$ has a multiple root if and only if $b^2 - 4ac = 0$. Give an alternative proof of the last statement using the fact that α is a multiple root of $f(x)$ if and only if $f(\alpha) = f'(\alpha) = 0$.

11. Let $f(x)$ be a nonzero polynomial with complex coefficients, m be a positive integer and α a complex number. Let $f^{(i)}(x)$ denotes the i-th derivative of $f(x)$, where by convention, $f^{(0)}(x) = f(x)$. Show that α is a root of $f(x)$ of multiplicity m if and only if

$$f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0 \text{ and } f^{(m)}(\alpha) \neq 0.$$

Deduce that the multiplicity of α as a root of $f(x)$ is the smallest nonnegative integer d such that $f^{(d)}(\alpha) \neq 0$.

12. Find the partial fraction decomposition of the following rational functions.

(i) $\dfrac{2}{x^2 - 2x - 3}$, (ii) $\dfrac{3-2x}{x^2-1}$, (iii) $\dfrac{x}{(x-2)(x-3)^3(2x+1)^2}$,

(iv) $\dfrac{3x^2+6x+4}{(x+3)(x^2+4)}$, (v) $\dfrac{9x}{(x^2-4x+5)(3x+1)}$, (v) $\dfrac{9x}{(x^2-4x+5)^3(3x+1)}$.

13. Find the 4-th term of $(2x-1)^{13}$.

14. Find the middle term of $(ax^4 - bx)^9$.

15. Given any integers n and k, let $\binom{n}{k}$ denote the binomial coefficient defined by

$$\binom{n}{k} = \begin{cases} \dfrac{n(n-1)\cdots(n-k+1)}{k!} & \text{if } k \geq 0, \\ 0 & \text{if } k = 0. \end{cases}$$

Prove that for any $n \in \mathbb{N}$, we have the following.

(i) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$, (ii) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$, (iii) $\displaystyle\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{2^{n+1}-1}{n+1}$,

(iv) $\displaystyle\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$, (v) $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$, (vi) $\displaystyle\sum_{k=0}^{n} k\binom{n}{k}^2 = \frac{(2n-1)!}{(n-1)!(n-1)!}$.