Indian Institute of Technology Bombay
MA 001 Preparatory Mathematics I

Autumn 2004 SRG

Exercise Set 4

1. If a cubic polynomial is monic and has $2, 2, 5$ as its roots, then find the coefficient of x^2.

2. Solve the equation $3x^3 - 16x^2 + 23x - 6 = 0$ if it is given that the product of two of its roots is 1.

3. The roots of the equation $9x^3 - 36x^2 + 44x - 16 = 0$ are in arithmetic progression. Find them.

4. The roots of the equation $3x^3 - 26x^2 + 52x - 24 = 0$ are in geometric progression. Find them.

5. If the cubic polynomial $x^3 + px + q$ has a multiple root, then what is the relation between p and q?
[Note: A *multiple root* is a root with multiplicity > 1.]

6. Find the sum of squares of the roots of the equation $x^3 - px^2 + qx - r = 0$.

7. Show that one root of the equation $x^3 + px^2 + qx + r = 0$ is the negative of another root if and only if $r = pq$.

8. Four persons enter a minibus which has 6 seats. In how many ways, can they take their places?

9. A cricket coach has 14 players in his coaching camp. How many different teams (of 11 players) can he form among these players? How many different teams can he form among these players if two teams are regarded different when the batting positions of the players has been changed?

10. How many code words can be made from the letters of the word STUPID if

    (i) the code words consist of 4 letters?

    (ii) the code words consist of 5 letters and must begin with a vowel?

11. How many different arrangements can be made by taking any 5 letters of the word EQUATION?

12. Consider the set $\{1, 2, \ldots, n\}$ with n elements. We know that the number of subsets of $\{1, 2, \ldots, n\}$ containing r elements is $\binom{n}{r}$. Now, find:

    (i) the number of subsets of $\{1, 2, \ldots, n\}$ containing r elements and always containing the number 1.

    (ii) the number of subsets of $\{1, 2, \ldots, n\}$ containing r elements and always containing the numbers 1 and 2.

    (iii) the number of subsets of $\{1, 2, \ldots, n\}$ containing r elements and always containing a fixed subset of $\{1, 2, \ldots, n\}$ with k elements.

13. Find the number of ways of choosing r unordered objects among n objects, if repetition is allowed. Equivalently, find the number of elements in the set

$$\{(j_1, j_2, \ldots, j_r) : j_1, j_2, \ldots, j_r \in \mathbb{Z} \text{ and } 1 \leq j_1 \leq j_2 \leq \cdots \leq j_r \leq n\}.$$

14. Find the number of monomials of degree r in n variables. Equivalently, find the number of elements in the set

$$\{(i_1, i_2, \ldots, i_n) : i_1, i_2, \ldots, i_n \in \mathbb{Z},\ i_1, i_2, \ldots, i_n \geq 0 \text{ and } i_1 + i_2 + \cdots + i_n = r\}.$$

15. If $f(x, y)$ is a homogeneous polynomial of degree d in two variables (with real or complex coefficients), then show that $f(x, y)$ can have at most d roots which are not proportional to each other.
    [Note: Two roots (α, β) and (α', β') are said to be *proportional* if $(\alpha, \beta) = (\lambda\alpha', \lambda\beta')$ for some $\lambda \neq 0$.]

16. If $f(x, y)$ is a homogeneous polynomial of degree d in two variables (with real or complex coefficients), then use the Fundamental Theorem of Algebra to show that $f(x, y)$ can be factored as a product of homogeneous linear polynomials (with complex coefficients), that is, we can write

$$f(x, y) = (\alpha_1 x + \beta_1 y)(\alpha_2 x + \beta_2 y) \cdots (\alpha_d x + \beta_d y),$$

    where $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_d, \beta_d)$ are pairs of complex numbers. Deduce that if (α, β) is a root of $f(x, y)$, then (α, β) is proportional to $(\beta_i, -\alpha_i)$ for some i, $1 \leq i \leq d$.

17. Give an example of a polynomial in two variables which has infinitely many roots that are not proportional to each other.

18. If $f(x, y)$ is a polynomial with coefficients in $\mathbb{R}$ such that $f(\alpha, \beta) = 0$ for all $(\alpha, \beta) \in \mathbb{R}^2$, then show that $f(x, y)$ must be the zero polynomial. Is this result true with $\mathbb{R}$ replaced by $\mathbb{C}$?

19. If $g(x, y)$ is a nonzero polynomial with coefficients in $\mathbb{R}$ and $f(x, y)$ is a polynomial with coefficients in $\mathbb{R}$ such that $f(\alpha, \beta) = 0$ for all $(\alpha, \beta) \in \mathbb{R}^2$ for which $g(\alpha, \beta) \neq 0$, then show that $f(x, y)$ must be the zero polynomial. Is this result true with $\mathbb{R}$ replaced by $\mathbb{C}$?