Indian Institute of Technology Bombay
MA 001 Preparatory Mathematics I

Autumn 2004 SRG

Exercise Set 7

1. Find the equation of the tangent to the following curves at the given point P.

   (i) $y = x^3 - x + 2,\ \ P = (1, 2)$, (ii) $x = y^2 + 2y + 1,\ \ P = (1, 0)$,
   (iii) $x^3 + y^3 + xy = 1,\ \ P = (1, 0)$, (iv) $x = 2t,\ y = t^2,\ \ P = (2, 1)$.

2. Suppose a curve C has a unique tangent at a point P. A vector $\mathbf{v}$ is said to be tangent to C at P if it is parallel to the line tangent to C at P. For each of the curves in Q. 1 above, find a unit vector tangent to the given curve at the given point.

3. Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are the vectors determined by the sides AB, BC and AC of a triangle ABC in the plane. Then express:

   (i) $\mathbf{w}$ in terms of $\mathbf{u}$ and $\mathbf{v}$, (ii) $\mathbf{v}$ in terms of $\mathbf{u}$ and $\mathbf{w}$.

4. Find the angle between $\mathbf{v}$ and $\mathbf{w}$, in each of the following cases.

   (i) $\mathbf{v} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{k}$ and $\mathbf{w} = 3\mathbf{k}$, (ii) $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = 2\mathbf{j} + 4\mathbf{k}$.

5. The projection of a vector $\mathbf{v} = OA$ onto a vector $\mathbf{w} = OB$ is the vector $\mathbf{p} = OP$ where P is the foot of the perpendicular from A onto the line OB. Suppose $\mathbf{v}$ and $\mathbf{w}$ are nonzero and θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Show that $\mathbf{p}$ has length $|\mathbf{v}| \cos\theta$ or $-|\mathbf{v}| \cos\theta$ according as θ is acute ($0 < \theta \le \pi/2$) or obtuse ($\pi/2 < \theta \le \pi$). Use this to show that
   $$\mathbf{p} = (|\mathbf{v}| \cos\theta)\, \frac{\mathbf{w}}{|\mathbf{w}|} = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\, \mathbf{w}.$$
   The projection of $\mathbf{v}$ onto $\mathbf{w}$ is sometimes denoted by $\mathrm{proj}_{\mathbf{w}}\mathbf{v}$, and its magnitude $|\mathbf{v}| \cos\theta$ is sometimes called the scalar component of $\mathbf{v}$ in the direction of $\mathbf{w}$.

6. Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors. Show that we can write $\mathbf{v} = \mathbf{p} + \mathbf{q}$, where $\mathbf{p}$ is a vector parallel to $\mathbf{w}$ and $\mathbf{q}$ is a vector orthogonal to $\mathbf{w}$. (Hint: Use $\mathrm{proj}_{\mathbf{w}}\mathbf{v}$.)

7. Determine the projection of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ onto $\mathbf{w} = 3\mathbf{i} - \mathbf{j}$. Moreover, write $\mathbf{v}$ as a sum of a vector parallel to $\mathbf{w}$ and a vector orthogonal to $\mathbf{w}$.

8. Find $\mathbf{v} \times \mathbf{w}$ and $\mathbf{w} \times \mathbf{v}$ if $\mathbf{v} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

9. Find a vector perpendicular to the plane spanned by $A = (1, -1, 0)$, $B = (2, 1, -1)$ and $C = (-1, 1, 2)$ in $\mathbb{R}^3$.

10. Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors in the 3-space and θ be the angle between them. Show that height of the paralleogram determined by $\mathbf{v}$ and $\mathbf{w}$ (with its base along $\mathbf{w}$) is equal to $|\mathbf{w}|\sin\theta$. Deduce that the area A of this parallelogram is the magnitude of the cross product of $\mathbf{v}$ and $\mathbf{w}$, that is $A = |\mathbf{v}\times\mathbf{w}|$.

11. Find the area of the paralleogram with vertices $A = (1, -1, 0)$, $B = (2, 1, -1)$ and $C = (-1, 1, 2)$. Also, find the area of the triangle ABC.

12. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be non-coplanar vectors in the 3-space and θ be the angle between $\mathbf{u}\times\mathbf{v}$ and $\mathbf{w}$. Show that the height of the parallelopiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ (and with its base along the plane spanned by $\mathbf{u}$ and $\mathbf{v}$) is equal to $|\mathbf{w}||\cos\theta|$. Deduce that the volume of this parallelopiped is the absolute value of scalar triple product $[\mathbf{u},\mathbf{v},\mathbf{w}] := (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$ of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

13. Find the volume of the box (parallelopiped) in 3-space determined by $\mathbf{u} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$, $\mathbf{v} = -2\mathbf{i}+3\mathbf{k}$ and $\mathbf{w} = 7\mathbf{j}-4\mathbf{k}$,

14. Let a matrix A be given by either of the following.

$$\text{(i)} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 4 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ -2 & -1 & -1 & 0 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} 1 & 4 & 7 & -3 \\ -2 & 3 & -6 & 1 \\ 0 & 11 & 8 & -5 \end{bmatrix}.$$

In each case, reduce A to a row-echelon form, find the rank of A, and solve the system of equations $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x}$ is a 4×1 column vector of variables, and $\mathbf{0}$ denotes the 4×1 zero vector.

15. Solve the following systems of equations by the Gauss Elimination Method.

$$\text{(i)} \begin{array}{rcl} x_1 + x_2 - x_3 + 2x_4 &=& 0 \\ 2x_2 + 3x_3 - x_4 &=& 0 \\ 2x_1 + 3x_2 + 3x_3 + 5x_4 &=& 0, \end{array} \quad \text{(ii)} \begin{array}{rcl} 2x_2 - 2x_3 + x_4 &=& 2 \\ 2x_1 - 8x_2 + 14x_3 - 5x_4 &=& 2 \\ x_1 + 3x_2 + +x_4 &=& 8. \end{array}$$

16. Determine the inverses of the following matrices, assuming $ad - bc \neq 0$ in (i).

$$\text{(i)} \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad \text{(iv)} \begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

17. Use Cramer's Rule to solve the following systems of equations.

$$\text{(i)} \begin{array}{rcl} 5x - 3y &=& 37 \\ 2x - 7y &=& 38, \end{array} \quad \text{(ii)} \begin{array}{rcl} x + 2y + 3z &=& 20 \\ 7x + 3y + z &=& 13 \\ x + 6 + 2z &=& 0, \end{array} \quad \text{(iii)} \begin{array}{rcl} 3x + 7y + 8z &=& -13 \\ 2x + 9z &=& -5 \\ -4x + y - 6z &=& 2. \end{array}$$