Indian Institute of Technology Bombay
MA 001 Preparatory Mathematics I

Mid-Semester Examination

Date: September 15, 2004 | Maximum Marks: 60
Duration: 2 hours | Weightage : 30 %

Note:

1. Attempt all the questions. The answers are to be written in the answer book provided to you.
2. Number the pages of the answer book as well as any supplements you may use.
3. Begin answer to each question on a new page. Write the page number against the corresponding question number on the index page. All the subparts of a question should be answered together.
4. Please make sure that your name and roll number appears on the answer book and on every supplement that you may use.

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1. Find all the square roots of
   (i) $\sqrt{1 - i\sqrt{3}}$, (ii) $25(-i)^{125}$. [6 points]
2. If z and w are complex numbers such that $|z| = 1$ and $|w| = 2$, then show that $|z - w| = |1 - \overline{z}w|$. [4 points]
3. Find the sum of all the primitive 6^{th} roots of unity. Also, find the product of all the primitive 6^{th} roots of unity. [6 points]
4. Suppose $f(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ is a quartic polynomial whose coefficients satisfy the identities
   $$a_0 + a_1 + a_2 + a_3 + a_4 = 0 \quad \text{and} \quad a_0 + a_2 + a_4 = a_1 + a_3.$$
   Show that $f(x)$ is divisible by $(x^2 - 1)$. [6 points]

5. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + \sin x$.

   (i) Does $f(x)$ have infinitely many roots in $\mathbb{R}$? Justify your answer.

   (ii) Is $f(x)$ a polynomial function? Justify your answer.

[8 points]

6. It is given that $x = 1$ is a root of the polynomial

$$f(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x + c.$$

   Determine the value of c. Moreover, determine the multiplicity of 1 as a root of $f(x)$. [6 points]

7. Find the partial fraction decomposition (with coefficients in $\mathbb{R}$) of the following rational functions.

   (i) $\dfrac{1}{x^2 + x^4}$, (ii) $\dfrac{6x^2 - 14x - 2}{x^3 + x^2 - 10x + 8}$. [8 points]

8. Prove that

$$\begin{aligned} 1 \; &+ \; \cos^5\left(\frac{\pi}{5}\right) + 5i\cos^4\left(\frac{\pi}{5}\right)\sin\left(\frac{\pi}{5}\right) - 10\cos^3\left(\frac{\pi}{5}\right)\sin^2\left(\frac{\pi}{5}\right) \\ &- \; 10i\cos^2\left(\frac{\pi}{5}\right)\sin^3\left(\frac{\pi}{5}\right) + 5\cos\left(\frac{\pi}{5}\right)\sin^4\left(\frac{\pi}{5}\right) + i\sin^5\left(\frac{\pi}{5}\right) = 0. \end{aligned}$$

[6 points]

9. Prove that for any positive integer n, we have

$$1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n}\binom{n}{n-1} + \frac{1}{n+1} = \frac{2^{n+1} - 1}{n+1}.$$

[4 points]

10. Prove that for any even positive integer $n = 2m$, we have

$$\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \cdots + (-1)^n\binom{n}{n}^2 = \frac{(-1)^m n!}{(m!)^2}.$$

[6 points]