Indian Institute of Technology Bombay
MA 001 Preparatory Mathematics I

Autumn 2012 SRG

Exercise Set 3

1. In each of the following, find the quotient $q(x)$ and the remainder $r(x)$ when the polynomial $f(x)$ is divided by the nonzero polynomial $g(x)$.

   (i) $f(x) = 2x^2 + 4x + 3$ and $g(x) = x^2 + 1$,
   (ii) $f(x) = x^2 + 4x + 3$ and $g(x) = 2x^2 + 1$,
   (iii) $f(x) = 3x^4 + 4x^2 + 3x + 5$ and $g(x) = 2x + 3$,
   (iv) $f(x) = 3x^4 + 4x^2 + 3x + 5$ and $g(x) = x^2 + 2$.

2. Find the remainder when the polynomial $x^{2004} + x^{2003} + x^{2002} + \ldots + x^2 + x + 1$ is divided by $x + 1$.

3. If $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ is a polynomial with complex coefficients such that $a_n = -(a_0 + a_1 + \cdots + a_{n-1})$, then prove that 1 is a root of $f(x)$.

4. Show that $\cos((x + 2)/3)$ is not a polynomial in x.

5. If $f(x) \in \mathbb{C}[x]$ is a nonconstant polynomial with complex coefficients and c is any complex number, then show that f takes the value c at only finitely many points, i.e., there are only finitely many $\alpha \in \mathbb{C}$ such that $f(\alpha) = c$. Can you put a bound on the number of $\alpha \in \mathbb{C}$ such that $f(\alpha) = c$?

6. If $f : \mathbb{R} \to \mathbb{R}$ is a periodic function, that is, if there is $p \in \mathbb{R}$ with $p \neq 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$, then show that $f(x)$ can not be a polynomial in x, unless f is a constant function. Does this result hold with $\mathbb{R}$ replaced by $\mathbb{C}$?

7. Show that $\sin x + \cos x$ is not a polynomial in x. Also show that $\sin(x/7) + \cos(x/5)$ is not a polynomial in x.

8. The complex exponential function e^z is defined by $e^z = e^x(\cos y + i \sin y)$, where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. Show that the complex exponential function e^z is not a polynomial in z.

9. A quotient of the form $p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials in $\mathbb{C}[x]$ with $q(x) \neq 0$, is called a rational function in x. Such a rational function in x gives rise to a complex valued function on any subset of $\mathbb{C}$ not containing the roots of $q(x)$. Show that $\sin x$, i.e, the sine function (from $\mathbb{R}$ to $\mathbb{R}$) is not a rational function in x.

10. If $f(x) = ax^2 + bx + c$ is a polynomial in $\mathbb{C}[x]$ of degree ≤ 2, then show that

$$4af(x) = (2ax + b)^2 - (b^2 - 4ac)$$

Use this to derive the formula for the roots of a quadratic, namely, if $a \neq 0$, then the roots of $f(x)$ are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and in particular, $f(x)$ has a multiple root if and only if $b^2 - 4ac = 0$. Give an alternative proof of the last statement using the fact that α is a multiple root of $f(x)$ if and only if $f(\alpha) = f'(\alpha) = 0$.

11. Let $f(x)$ be a nonzero polynomial with complex coefficients, m be a positive integer and α a complex number. Let $f^{(i)}(x)$ denotes the i-th derivative of $f(x)$, where by convention, $f^{(0)}(x) = f(x)$. Show that α is a root of $f(x)$ of multiplicity m if and only if

$$f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0 \text{ and } f^{(m)}(\alpha) \neq 0.$$

Deduce that the multiplicity of α as a root of $f(x)$ is the smallest nonnegative integer d such that $f^{(d)}(\alpha) \neq 0$.

12. The Fundamental Theorem of Algebra states that every nonconstant polynomial with coefficients in $\mathbb{C}$ has at least one root in $\mathbb{C}$. Assuming this result, prove the following:

> If $p(X)$ is a nonzero polynomial of degree n with coefficients in $\mathbb{C}$, then $p(X)$ has exactly n roots in $\mathbb{C}$, counting multiplicities. More precisely, there exist distinct complex numbers $\alpha_1, \ldots, \alpha_h$ and positive integers $e_1, \ldots, e_h$ such that
>
> $$p(X) = c(X - \alpha_1)^{e_1} \cdots (X - \alpha_h)^{e_h}$$
>
> for some $c \in \mathbb{C}$ with $c \neq 0$.

13. If $f(x, y)$ is a homogeneous polynomial of degree d in two variables (with real or complex coefficients), then show that $f(x, y)$ can have at most d roots which are not proportional to each other.
[Note: Two roots (α, β) and (α', β') are said to be *proportional* if $(\alpha, \beta) = (\lambda\alpha', \lambda\beta')$ for some $\lambda \neq 0$.]

14. If $f(x, y)$ is a homogeneous polynomial of degree d in two variables (with real or complex coefficients), then use the Fundamental Theorem of Algebra to show that $f(x, y)$ can be factored as a product of homogeneous linear polynomials (with complex coefficients), that is, we can write

$$f(x, y) = (\alpha_1 x + \beta_1 y)(\alpha_2 x + \beta_2 y) \cdots (\alpha_d x + \beta_d y),$$

where $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_d, \beta_d)$ are pairs of complex numbers. Deduce that if (α, β) is a root of $f(x, y)$, then (α, β) is proportional to $(\beta_i, -\alpha_i)$ for some i, $1 \leq i \leq d$.

15. Give an example of a polynomial in two variables which has infinitely many roots that are not proportional to each other.

16. If $f(x, y)$ is a polynomial with coefficients in $\mathbb{R}$ such that $f(\alpha, \beta) = 0$ for all $(\alpha, \beta) \in \mathbb{R}^2$, then show that $f(x, y)$ must be the zero polynomial. Is this result true with $\mathbb{R}$ replaced by $\mathbb{C}$?

17. If $g(x, y)$ is a nonzero polynomial with coefficients in $\mathbb{R}$ and $f(x, y)$ is a polynomial with coefficients in $\mathbb{R}$ such that $f(\alpha, \beta) = 0$ for all $(\alpha, \beta) \in \mathbb{R}^2$ for which $g(\alpha, \beta) \neq 0$, then show that $f(x, y)$ must be the zero polynomial. Is this result true with $\mathbb{R}$ replaced by $\mathbb{C}$?