Indian Institute of Technology Bombay

MA 103 : Mathematics I
Autumn 2001

Tutorial Sheet No. 1

Part A

1. Prove the Binomial Theorem (for positive integral exponents). In other words, prove that for any $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$, we have
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$
Recall that for any $a \in \mathbb{R}$ and $k \in \mathbb{Z}$, the binomial coefficient associated to a and k is defined by
$$\binom{a}{k} = \begin{cases} \dfrac{a(a-1)\cdots(a-k+1)}{k!} & \text{if } k \geq 0 \\ 0 & \text{if } k < 0. \end{cases}$$
where for $k \in \mathbb{N}$, $k!$ (read as: k factorial) denotes the product of the first k positive integers. We usually set the empty product (and in particular, $0!$) to be 1. Thus, for example, if $k = 0$, then the above binomial coefficient equals 1.

2. Let S be a subset of $\mathbb{R}$ If S is bounded above and $M \in \mathbb{R}$, then show that $M = \sup S$ if and only if M is an upper bound of S and for every $\epsilon > 0$, there exists $x \in S$ such that $M - \epsilon < x \leq M$. Likewise, if S is bounded below and $m \in \mathbb{R}$, then show that $m = \inf S$ if and only if m is a lower bound of S and for every $\epsilon > 0$, there exists $x \in S$ such that $m \leq x < m + \epsilon$.

3. If $a, b \in \mathbb{R}$ with $a < b$, then show that there exist infinitely many rationals as well as infinitely many irrationals between a and b.

4. Let $n \in \mathbb{N}$ and $x_1, \ldots, x_n$ be positive real numbers. Prove that
$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n} \geq n \left[\frac{1}{x_1} + \ldots + \frac{1}{x_n} \right]^{-1}.$$
[The above result is sometimes called the A.M.-G.M.-H.M. Inequality and the number $n \left[x_1^{-1} + \cdots + x_n^{-1} \right]^{-1}$ is called the harmonic mean of $x_1, \ldots, x_n$.]

5. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by either of the following.
$$\text{(i) } f(x) = 3x + 5, \quad \text{(ii) } f(x) = 2x^2 + 5x + 2, \quad \text{(iii) } f(x) = x^2 + 1.$$
Determine whether or not f is (i) one-one (ii) onto (iii) bijective. In case f is one-one, find the inverse of f. In case f is not onto, find the range of f. Also, sketch the graph of f.

6. Prove that the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x} + \sqrt{2x}$ is an algebraic function.

Part B

1. Let S be a subset of $\mathbb{R}$ and $c \in \mathbb{R}$. Define the additive translate $c + S$ and the multiplicative translate cS of S as follows.

$$c + S = \{c + x : x \in S\} \quad \text{and} \quad cS = \{cx : x \in S\}.$$

If S is bounded, then show that $c + S$ and cS are bounded and moreover,

$$\sup(c + S) = c + \sup S \quad \text{and} \quad \sup(cS) = \begin{cases} c \sup S & \text{if } c \geq 0 \\ c \inf S & \text{if } c \leq 0. \end{cases}$$

2. Given any $x, y \in \mathbb{R}$ with $x \neq y$, show that there exists $\delta > 0$ such that the intervals $(x - \delta, x + \delta)$ and $(y - \delta, y + \delta)$ have no point in common.

3. Let $n, m \in \mathbb{N}$ be such that $m \geq n$.

   (i) If $x_1, \ldots, x_m$ are real numbers such that $0 \leq x_1 \leq \cdots \leq x_m$, then show that the arithmetic mean of $x_1, \ldots, x_n$ is less than or equal to the arithmetic mean of $x_1, \ldots, x_m$. (Hint: Induct on m.)

   (ii) If $x \in \mathbb{R}$ is such that $x \geq 1$, then show that $\dfrac{x^m - 1}{m} \geq \dfrac{x^n - 1}{n}$.

4. Prove the following version of Binomial Inequality for rational exponents: if $x \in \mathbb{R}$ and $r \in \mathbb{Q}$ are such that $x \geq 0$ and $r \geq 1$, then $(1 + x)^r \geq 1 + rx$. (Hint: Use Exercise 3.)

5. Given any $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$, define functions g_c, h_c, k_c and ℓ_c from $\mathbb{R}$ to $\mathbb{R}$ as follows.

$$g_c(x) = f(x) + c, \quad h_c(x) = cf(x), \quad k_c(x) = f(x + c), \quad \ell_c(x) = f(cx).$$

If f is given by $f(x) = x^n$ for all $x \in \mathbb{R}$, then sketch the graph of g_c, h_c, k_c and ℓ_c when $n = 1, 2$ or 3 and $c = 0, 1, 2, -1, -2, 1/2$ or $-1/2$.

6. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the function defined by either of the following.

$$\text{(i) } f(x) = 1/x^2, \quad \text{(ii) } f(x) = 1/|x|, \quad \text{(iii) } f(x) = 1/\sqrt{|x|}.$$

Determine whether or not f is (i) one-one (ii) onto (iii) bijective. In case f is one-one, find the inverse of f. In case f is not onto, find the range of f. Also, sketch the graph of f.

7. Prove that the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x} + \sqrt[3]{x}$ is an algebraic function.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be the absolute value function defined by $f(x) = |x|$. Show that f is not a rational function.