Indian Institute of Technology Bombay

MA 103 : Mathematics I
Autumn 2001

Tutorial Sheet No. 2

Part A

1. Let $f : [0, \infty) \to \mathbb{R}$ be the function defined by $f(x) = \sqrt{x}$. Show that f is strictly increasing and strictly concave on $[0, \infty)$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by either of the following.

   (i) $f(x) = x^3$, (ii) $f(x) = |x|$.

   Sketch the graph of f and determine the points of local extremum as well as the points of inflection, if any, in each case.

3. Give examples of functions $f : (0, 1) \to \mathbb{R}$ such that f is

   (i) strictly increasing and strictly convex

   (ii) strictly increasing and strictly concave

   (iii) strictly decreasing and strictly convex

   (iv) strictly decreasing and strictly concave.

4. Which of the following sequences are bounded? Which of them are convergent? In case of convergence, find the limit.

   (i) $a_n := \dfrac{1}{n^2}$, (ii) $a_n := \sqrt{n}$, (iii) $a_n := \dfrac{n}{2n+1}$,

   (iv) $a_n := \sqrt{n}(\sqrt{n+1} - \sqrt{n})$.

5. Let (a_n) and (b_n) be sequences in $\mathbb{R}$. Under which of the following conditions is the sequence $(a_n b_n)$ convergent? Justify.

   (i) (a_n) is convergent.

   (ii) (a_n) is convergent and (b_n) is bounded.

   (iii) (a_n) converges to 0 and (b_n) is bounded.

   (iv) (a_n) and (b_n) are convergent.

6. Show that the following sequences are convergent and find $\lim_{n\to\infty} a_n$.

   (i) $a_1 := 1$ and $a_{n+1} := \sqrt{2 + a_n}$ for $n \in \mathbb{N}$,

   (ii) $a_1 := 1$ and $a_{n+1} := \dfrac{a_n}{2a_n + 1}$ for $n \in \mathbb{N}$.

7. Show that $(n!)^{1/n} \to \infty$.

Part B

1. Consider $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \lfloor x \rfloor \quad \text{and} \quad g(x) = \lceil x \rceil$$

where for $x \in \mathbb{R}$, $\lfloor x \rfloor = [x]$ denotes the largest integer $\leq x$ whereas $\lceil x \rceil$ denotes the smallest integer $\geq x$ ($\lceil x \rceil$ is sometimes called the ceiling of x). Sketch the graphs of f and g, and find the points of local extremum of these two functions.

2. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the function defined by $f(x) = 1/x$. Show that f is strictly decreasing and strictly convex on $(0, \infty)$, and it is strictly decreasing and strictly concave on $(-\infty, 0)$.

3. Give an example of a nonconstant function $f : (-1, 1) \to \mathbb{R}$ such that f has a local extremum at 0 as well as a point of inflection at 0.

4. Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions and define $h(x) = f(x)g(x)$ for $x \in \mathbb{R}$. Which of the following statements are true? Why?

   (i) If $f(x) \geq 0$, $g(x) \geq 0$ for all $x \in \mathbb{R}$, and if f as well as g has a local maximum at $x = c$, then so does h.

   (ii) If f as well as g has a point of inflection at $x = c$, then so does h.

5. Which of the following sequences are bounded? Which of them are convergent? In case of convergence, find the limit.

   (i) $a_n := \dfrac{1}{n^3}$, (ii) $a_n := n^{1/3}$, (iii) $a_n := \dfrac{3n^2}{n^2 + n + 1}$,

   (iv) $a_n := n^{3/2}(\sqrt{n^3 + 1} - \sqrt{n^3})$.

6. Establish the following limits.

   (i) $\lim\limits_{n\to\infty} n^k x^n = 0$ if $k \in \mathbb{N}$, $x \in \mathbb{R}$ and $|x| < 1$,

   (ii) $\lim\limits_{n\to\infty} n^{1/n} = 1$.

7. Show that the following sequences are convergent and find $\lim\limits_{n\to\infty} a_n$.

   (i) $a_1 := 1$ and $a_{n+1} := \dfrac{3a_n + 2}{6}$ for $n \in \mathbb{N}$,

   (ii) $a_1 := 2$ and $a_{n+1} := \sqrt{1 + a_n}$ for $n \in \mathbb{N}$.