Indian Institute of Technology Bombay

MA 103 : Mathematics I
Autumn 2001

Tutorial Sheet No. 3

Part A

1. Does there exist a function $f[1,3] \to \mathbb{R}$ which is continuous at 2 and which satisfies one of the following conditions.

   (i) $f(x) = (x^3 - 3x - 2)/(x-2)$, if $x \neq 2$,

   (ii) $f(x) = \begin{cases} x, & \text{if } x \in [1,2), \\ x/2, & \text{if } x \in (2,3], \end{cases}$

   (iii) $f((2n-1)/n) = (-1)^n$ for all $n \in \mathbb{N}$,

   (iv) $f((2n+1)/n) = 2^{1/n}$ for all $n \in \mathbb{N}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x)f(y)$ for all $x,\ y \in \mathbb{R}$. If f is continuous at 0, then show that f is continuous at every $c \in \mathbb{R}$. (Do you know an example of such a function?)

3. If $g : [a,b] \to [a,b]$ is continuous, show that there is some $c \in [a,b]$ such that $g(c) = c$. (Hint: IVP)

4. Analyze the following functions for uniform continuity.

   (i) $f(x) = 1/x,\ x \in (0,1]$ (ii) $f(x) = x^2,\ x \in (0,1)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. If $\lim\limits_{x \to c} f(x)$ exists, then show that

$$\lim_{h \to 0^+} [f(c+h) - f(c-h)] = 0.$$

   Is the converse true? Justify your answer.

6. Given $\epsilon > 0$, find $\delta > 0$ such that $|f(x) - \ell| < \epsilon$ whenever $0 < |x-c| < \delta$, if $f(x) = x^2 + 1,\ c = 1$ and $\ell = 2$.

7. Find the asymptotes of the following curves.

   (i) $y = \dfrac{x^2}{x^2+1}$ (ii) $y = \dfrac{x^2+1}{x},\ x \neq 0$.

Part B

1. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, then show that (i) f is continuous at every $c \in \mathbb{R}$ and (ii) $f(sx) = sf(x)$ for all $s, x \in \mathbb{R}$. Deduce that there exists $r \in \mathbb{R}$ such that $f(x) = rx$ for all $x \in \mathbb{R}$.

2. Let $f(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 1-x, & \text{if } x \text{ is irrational.} \end{cases}$

   Show that f is continuous only at $1/2$.

3. Let $f : [a,b] \to \mathbb{R}$ be continuous. Given $c_1, \ldots, c_n$ in $[a,b]$, show that there is $c \in [a,b]$ such that $f(c) = \dfrac{f(c_1) + \cdots + f(c_n)}{n}$. (Hint: IVP)

4. If a function f satisfies one of the following conditions, can it be continuous? Why?

   (i) $f : [1,10] \to \mathbb{R},\ f(1) = 0,\ f(10) = 11,$
   range of $f \subset [-1,0] \cup [1,11]$

   (ii) $f : [0,1] \to \mathbb{R}$ and range of $f = (-1,1)$

   (iii) $f : [-1,1] \to \mathbb{R}$ and range of $f = [0,\infty)$.

5. Analyze the following functions for uniform continuity.
   (i) $f(x) = x,\ x \in \mathbb{R}$ (ii) $f(x) = \sqrt{1-x^2},\ x \in [-1,1]$.

6. Show that $f(x) \to \infty$ as $x \to \infty$ if (i) $f(x) = \dfrac{3x^2+1}{2x+1}$, (ii) $f(x) = [x]$.

7. Given $\epsilon > 0$, find $\delta > 0$ such that $|f(x) - \ell| < \epsilon$ whenever $0 < |x - c| < \delta$ if $f(x) = \dfrac{1}{x}$, $c \neq 0$ and $\ell = \frac{1}{c}$.

8. Find the asymptotes of the following curves.
   (i) $y = \dfrac{x^2}{x^2-1},\ x \neq \pm 1$ (ii) $y = \dfrac{x^2+x-2}{x-2},\ x \neq 2$.

9. Give an example to show that an increasing function defined on an interval I need not be continuous. However, show that a convex function defined on an open interval I is continuous on I.