Indian Institute of Technology Bombay

MA 103 : Mathematics I
Autumn 2001

Tutorial Sheet No. 5

Part A

1. Let $f : (0, \infty) \to \mathbb{R}$ be such that $f(xy) = f(x) + f(y)$ for all $x, y \in (0, \infty)$ and f is differentiable at 1. Show that $f(x) = \beta \ln x$ for all $x \in (0, \infty)$, where $\beta = f^{'}(1)$.

2. The hyperbolic sine and cosine functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad \text{for} \quad x \in \mathbb{R}.$$

Show that for any $t \in \mathbb{R}$, $(\cosh t, \sinh t)$ is a point on the hyperbola $x^2 - y^2 = 1$. Also, show that $(\sinh)^{'}x = \cosh x$ and $(\cosh)^{'}x = \sinh x$ for $x \in \mathbb{R}$. Further, show that for $x, y \in \mathbb{R}$, we have $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$ and $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$. Sketch a graph of these functions.

3. Evaluate the following limits.
(i) $\displaystyle\lim_{x\to\infty} \frac{5x^2 - 3x}{7x^2 + 1}$ (ii) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1 + (x^2/2)}{x^4}$ (iii) $\displaystyle\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x}$.

4. Show that if $f, g : \mathbb{R} \to \mathbb{R}$ are functions defined by

$$f(x) = \begin{cases} x + 2, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x + 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0, \end{cases}$$

then $\displaystyle\lim_{x\to 0} \frac{f^{'}(x)}{g^{'}(x)} = 1$ but $\displaystyle\lim_{x\to 0} \frac{f(x)}{g(x)} = 2$. Does this contradict L'Hôpital's Rule?

5. Find the maximum value of $f(x) = (\sin x - \cos x)^2,\ x \in \mathbb{R}$.

6. Find the absolute minimum of the function $f(x) = e^x - 1 - x,\ x \in \mathbb{R}$. Show that $e^x > 1 + x$ for all $x \in \mathbb{R},\ x \neq 0$. Determine which of the two numbers e^π and π^e is greater. (Hint : Let $x = (\pi/e) - 1$.)

7. A post office will accept a box for shipment only if the sum of its length and its girth (that is, distance around) does not exceed 84 inches. Find the dimensions of the largest acceptable box with a square end.

Part B

1. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ and f is differentiable at 0. Show that either $f(x) = 0$ for all $x \in \mathbb{R}$ or $f(x) = e^{\alpha x}$ for all $x \in \mathbb{R}$, where $\alpha = f'(0)$.

2. Find the nth Taylor polynomial around 0 for

   (i) $f(x) = \ln(1+x)$ (ii) $f(x) = e^x$

   (iii) $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$.

3. Evaluate the following limits.

   (i) $\lim_{x\to\infty} \frac{\sqrt{x+2}}{\sqrt{x+1}}$ (ii) $\lim_{x\to\infty} x^{1/x}$ (iii) $\lim_{x\to 0+} x \ln x$

4. Consider the following application of L'Hôpital's Rule :

$$\lim_{x\to 1} \frac{3x^2 - 2x - 1}{x^2 - x} = \lim_{x\to 1} \frac{6x-2}{2x-1} = \lim_{x\to 1} \frac{6}{2} = 3.$$

   Is it correct? Justify.

5. Let $f(x) = \dfrac{1}{x-1}$ for $x \neq 1$ and $g(x) = x$. Show that as $x \to 1^+$, $f'(x)/g'(x) \to -\infty$, but $f(x)/g(x) \to \infty$. Does this contradict L'Hôpital's Rule? Justify.

6. Order the following functions in the descending rate of growth, that is, from the fastest growing to the slowest growing :

$$2^x,\ e^x,\ x^x,\ (\ln x)^x, e^{x/2},\ x^{1/2},\ \log_2 x,\ \ln(\ln x),\ (\ln x)^2,\ x^e,\ x^2,\ \ln x,\ (2x)^x,\ x^{2x}.$$

7. Let $f(x) = 1 + 12|x| - 3x^2$. Find the absolute maximum and the absolute minimum of f on $[-2, 5]$. Verify it from the sketch of the curve $y = f(x)$ on $[-2, 5]$.

8. A window is to be made in the form of a rectangle surmounted by a semicircular portion with diameter equal to the base of the rectangle. The rectangular portion is to be of clear glass and the semicircular portion is to be of coloured glass admitting only half as much light per square foot as the clear glass. If the total perimeter of the window frame is to be p feet, find the dimensions of the window which will admit the maximum light.

9. The stiffness of a rectangular beam is proportional to the product of its breadth and the cube of its thickness but is not related to its length. Find the proportions of the stiffest beam that can be cut from a cylindrical log of diameter d.