Indian Institute of Technology Bombay

MA 103 : Mathematics I
Autumn 2001

Tutorial Sheet No. 6

Part A

1. Find the local maxima and the local minima of the function $f(x) = x^m(1-x)^n$, $x \in [0,1]$, where m and n are positive integers.

2. Sketch the following curves after locating intervals of increase/decrease, intervals of convexity/concavity, points of local maxima/minima and points of inflection. How many times and approximately where does the curve cross the x-axis?

   (i) $y = 2x^3 + 2x^2 - 2x - 1$ (ii) $y = x + \sin x$.

3. Find the linear approximation near 0 of the function f defined by $f(x) = \dfrac{1}{1-x}$, $x \neq 1$ Find an estimate for the error $e_1(x)$ when $x > 0$ and when $x < 0$. Find an upper bound for the error $e_1(x)$ which is valid for all $x \in (0, 0.1)$, and also one which is valid for all $x \in (-0.1, 0)$.

4. (i) Find an approximate value of $\sqrt{3}$ by using the linear approximation of $g(x) = \sqrt{x}$ for x near 4.

   (ii) Let $f(x) = \sqrt{x} + \sqrt{x+1} - 4$. Show that there is a unique $x_0 \in (3,4)$ such that $f(x_0) = 0$. Using the linear approximation of f near 3, find an approximation x_1 of x_0. Find x_0 exactly and determine the error $|x_1 - x_0|$.

5. For the function $f : [0,2] \to \mathbb{R}$ defined by

$$f(x) = x - 1 - \frac{\sin x}{2}, \quad x \in [0,2],$$

   show that the equation $f(x) = 0$ has a unique root in the interval $[0,2]$. Use Newton's method starting with $x_0 = 1.5$ to find an approximate value of this root. (You may use a calculator.)

6. Let $f(x) = \sqrt{x-1}$ if $x \geq 1$ and $f(x) = -\sqrt{1-x}$ if $x < 1$. If $x_0 \neq 1$, show that the Newton sequence for f oscillates between x_0 and $2 - x_0$.

7. Show that each of the following functions maps the interval I mentioned against it into itself and has a unique fixed point in that interval. Also, show that if x_0 belongs to this interval, then the Picard sequence converges to this unique fixed point.

   (i) $g(x) = \cos x$, $I = [0,1]$ (ii) $g(x) = 1 + \dfrac{\sin x}{2}$, $I = [0,2]$

Part B

1. For which constants a, b, c, d does the function $f(x) = ax^3 + bx^2 + cx + d,\ x \in \mathbb{R}$, have a local maximum at -1, a point of inflection at 1 and satisfies $f(-1) = 10,\ f(1) = -6$?

2. Sketch the following curves after locating intervals of increase/decrease, intervals of convexity/concavity, points of local maxima/minima and points of inflection. How many times and approximately where does the curve cross the x-axis?

   (i) $y = x^3 - 6x^2 + 9x + 1$ (ii) $y = \dfrac{x^2}{x^2+1}$

3. Let $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$ for $x > 0$. Write down the linear and the quadratic approximations $L(x)$ and $Q(x)$ of $f(x)$ near 4. Find the errors $f(4.41) - L(4.41)$ and $f(4.1) - Q(4.41)$.

4. Sketch a continuous curve $y = f(x)$ having the following properties :

   $f(-2) = 8,\ f(0) = 4,\ f(2) = 0;\ \ f'(2) = f'(-2) = 0;$

   $f'(x) > 0$ for $|x| > 2,\ f'(x) < 0$ for $|x| < 2;$

   $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$.

5. Find an approximate value of $(8.01)^{4/3} + (8.01)^2 - (8.01)^{-1/3}$.

6. Find the linear approximation near 0 of the the function f defined by

   $$f(x) = \sqrt{1+x}, \quad x \geq -1$$

   Find an estimate for the error $e_1(x)$ when $x > 0$ and when $x < 0$. Find an upper bound for the error $e_1(x)$ which is valid for all $x \in (0, 0.1)$, and also one which is valid for all $x \in (-0.1, 0)$.

7. Let $f(x) = (x-1)^{1/3}$ for $x \in \mathbb{R}$. If $x_0 \neq 1$, show that the Newton sequence for f is unbounded.

8. Let $f(x) = x^2 - 2x - 3$ for $x \in \mathbb{R}$. If $x_0 < 1$, prove that the Newton sequence converges to the root -1 of $f(x) = 0$ and if $x_0 > 1$, then the Newton sequence for f converges to the other root 3.

9. Show that each of the following functions maps the interval I mentioned against it into itself and has a unique fixed point in that interval. Also, show that if x_0 belongs to this interval, then the Picard sequence converges to this unique fixed point.

   (i) $g(x) = \sqrt{\sin x},\ \ I = [\pi/4, \pi/2]$ (ii) $g(x) = \dfrac{1}{1+x^2},\ \ I = [0, 1]$.