Indian Institute of Technology Bombay

MA 103 : Mathematics I
Autumn 2001

Tutorial Sheet No. 7

Part A

1. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \text{ is rational} \\ 0 & \text{if } x \in [0,1] \text{ is irrational.} \end{cases}$$

Show that f is bounded but not Riemann integrable on $[0,1]$.

2. Let $f : [0,3] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } \ 0 \le x \le 1 \\ 2, & \text{if } \ 1 < x \le 2 \\ -1, & \text{if } \ 2 < x \le 3. \end{cases}$$

Show that f is neither continuous nor montonic on [0,3], but f is Riemann integrable on $[0,3]$. Find $\displaystyle\int_0^3 f(x)dx$.

3. Assuming that f is Riemann integrable on a suitable interval, express

$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right]$$

as a Riemann integral. Use this to evaluate $\lim\limits_{n\to\infty} S_n$ in each of the following cases.

(i) $S_n = \dfrac{1}{n^{17}}\displaystyle\sum_{i=1}^{n} i^{16}$ (ii) $S_n = \dfrac{1}{n^{5/2}}\displaystyle\sum_{i=1}^{n} i^{3/2}$ (iii) $S_n = \displaystyle\sum_{i=1}^{n} \frac{1}{\sqrt{in+n^2}}$.

4. Let $f(x) \ge 0$ for all $x \in [0,\infty)$ and f be continuous on $[0,b]$ for each $b \ge 0$. If the area bounded by the x-axis, the lines $x = 0, x = b$ and the curve $y = f(x)$ is given by $\sqrt{b^2+1} - 1$ for each $b > 0$, determine the function f.

5. Let $f : [a,\infty) \to \mathbb{R}$ be a bounded function such that f is Riemann integrable on $[a,x]$ for every $x \ge a$. Let $F(x) = \displaystyle\int_a^x f(t)dt$ for $x \ge a$. Show that F is uniformly continuous on $[a,\infty)$.

6. Let p be a real number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral $\int_a^{a+p} f(t)dt$ has the same value for every real number a.

   (Hint : $F^{'}(a) = 0$, where $F(a) = \int_a^{a+p} f(t)dt,\ a \in \mathbb{R}$.)

7. Evaluate $\lim\limits_{h\to 0} \frac{1}{h} \int_x^{x+h} \frac{du}{u+\sqrt{u^2+1}}$.

Part B

1. Let $f(x) = 1$ if $x \in [0,1]$ and $f(x) = 2$ if $x \in (1,2]$. Show from the first principles that f is Riemann integrable on $[0,2]$ and find $\int_0^2 f(x)dx$.

2. Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a,b]$. Show that $\int_a^b f(x)dx \geq 0$. If, in fact, f is continuous and $\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a,b]$. Give an example of a Riemann integrable function on $[a,b]$ such that $f(x) \geq 0$ for all $x \in [a,b]$ and $\int_a^b f(x)dx = 0$, but $f(x) \neq 0$ for some $x \in [a,b]$.

3. Let $n, m \in \mathbb{N}$. Find $\lim\limits_{m\to\infty} \int_0^1 \frac{x^n}{(1+x)^m}dx$ and $\lim\limits_{n\to\infty} \int_0^1 \frac{x^n}{(1+x)^m}dx$.

4. Prove that $2\sin\frac{h}{2}[\sin h + \sin 2h + \cdots + \sin mh] = \cos\frac{h}{2} - \cos(m+\frac{1}{2})h$ for all $h \in \mathbb{R}$ and $m = 1, 2, \ldots$. Hence find $\int_0^{\pi/2} \sin x\, dx$ from the first principles.

5. For $n \in \mathbb{N}$, let S_n be given by the following. In each case, evaluate $\lim\limits_{n\to\infty} S_n$.

   (i) $S_n = \sum\limits_{i=1}^{n} \frac{n}{i^2+n^2}$ (ii) $S_n = \frac{1}{n}\sum\limits_{i=1}^{n} \cos\frac{i\pi}{n}$

   (iii) $S_n = \frac{1}{n}\left\{\sum\limits_{i=1}^{n}\left(\frac{i}{n}\right) + \sum\limits_{i=n+1}^{2n}\left(\frac{i}{n}\right)^{3/2} + \sum\limits_{i=2n+1}^{3n}\left(\frac{i}{n}\right)^{2}\right\}$.

6. Find an approximate value of $1^{1/3} + 2^{1/3} + \cdots + 1000^{1/3}$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R},\ \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda}\int_0^x f(t)\sin\lambda(x-t)dt.$$

   Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g'(0)$.