Indian Institute of Technology Bombay

MA 103 : Mathematics I
Autumn 2001

Tutorial Sheet No. 8

Part A

1. Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that for every $x \in [a, b]$,
$$\int_a^x \left[\int_a^u f(t)dt \right] du = \int_a^x (x-u) f(u) du.$$

2. Let $n \in \mathbb{N}$ and $f : [a, b] \to \mathbb{R}$ be such that $f', f'', \ldots, f^{(n+1)}$ exist and are continuous on $[a, b]$. Show that
$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{1}{n!}\int_a^b (b-t)^n f^{(n+1)}(t)dt.$$
[This result is an alternate version of Taylor's Theorem.]

3. If $f : [0, 2\pi] \to \mathbb{R}$ is given by a *trigonometric polynomial*, that is, if
$$f(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$
where $a_0, a_1, \ldots, a_n, b_1, \ldots, b_n$ are some fixed real numbers, then show that: (i) $\displaystyle\int_0^{2\pi} f(x)\,dx = 2\pi a_0$ (ii) $\displaystyle\int_0^{2\pi} f(x)\cos mx\,dx = \pi a_m$, and (iii) $\displaystyle\int_0^{2\pi} f(x)\sin mx\,dx = \pi b_m$, where $1 \le m \le n$.

4. Find $\dfrac{dF}{dx}$, if for $x \in \mathbb{R}$ (i) $F(x) = \displaystyle\int_1^{2x} \cos(t^2)dt$ (ii) $F(x) = \displaystyle\int_0^{x^2} \cos(t)dt$.

5. (i) Show that for any $n \in \mathbb{N}$, $\displaystyle\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$.

   (ii) Show that for any $m \in \mathbb{N}$,
$$\int_0^{\pi/2} \sin^{2m} x\,dx = \frac{(2m-1)(2m-3)\cdots 3\cdot 1}{(2m)(2m-2)\cdots 4\cdot 2}\cdot\frac{\pi}{2} = \frac{(2m)!}{[2^m m!]^2}\cdot\frac{\pi}{2}$$

and

$$\int_0^{\pi/2} \sin^{2m+1} x\,dx = \frac{2m(2m-2)\cdots 4\cdot 2}{(2m+1)(2m-1)\cdots 3\cdot 1} = \frac{[2^m m!]^2}{(2m+1)!}.$$

(iii) For $m \in \mathbb{N}$, let $\mu_m = \int_0^{\pi/2} \sin^{2m}\,dx \Big/ \int_0^{\pi/2} \sin^{2m+1} x\,dx$. Show that $1 \le \mu_m \le \dfrac{2m+1}{2m}$ and hence that $\mu_m \to 1$ as $m \to \infty$. (Hint : $\sin^{2m+1} x \le \sin^{2m} x \le \sin^{2m-1} x$ for all $x \in [0, \pi/2]$.)
Deduce that

$$\sqrt{\pi} = \lim_{m\to\infty} \frac{(m!)^2 2^{2m}}{(2m)!\,\sqrt{m}}.$$

[This is known as Wallis' Formula. It can be used to compute approximations for π.]

Part B

1. Find $\lim\limits_{n\to\infty} \int_0^1 \dfrac{nx^{n-1}}{1+x} dx$. (Hint : Integration by parts)

2. Given $m, n \in \mathbb{N}$, show that $\int_0^{\pi} x^m (1-x)^n\,dx = \dfrac{m!\,n!}{(m+n+1)!}$.

3. Show that $\int_0^{\pi} \dfrac{\sin[n + (x/2)]}{\sin(x/2)}\,dx = \pi$.

   (Hint : The integrand equals $\dfrac{1}{2} + \sum\limits_{k=1}^{n} \cos kx$.)

4. Evaluate (i) $\lim\limits_{x\to 0} \dfrac{1}{x^3} \int_0^x \dfrac{t^2 dt}{t^4+1}$ (ii) $\lim\limits_{x\to 0} \dfrac{1}{x^6} \int_0^{x^2} \dfrac{t^2 dt}{t^6+1}$.

5. In each case, compute $f(2)$ if f is continuous and satisfies the given formula for all $x \ge 0$: (i) $\int_0^{x^2} f(t)dt = x^2(1+x)$ (ii) $\int_0^{x^2(1+x)} f(t)dx = x$.

6. (i) Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Show that $\int_0^a (a^2 - x^2)^n\,dx = \dfrac{2^{2n}(n!)^2}{(2n+1)!} \cdot a^{2n+1}$.

   (ii) Deduce that

$$1 - \frac{1}{3}\binom{n}{1} + \frac{1}{5}\binom{n}{2} - \frac{1}{7}\binom{n}{3} + \cdots + \frac{(-1)^n}{2n+1}\binom{n}{n} = \frac{(2^n n!)^2}{(2n+1)!}.$$