MA 105 : Calculus

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Basic Information

- Lectures: Tue, Thu 11 am – 12.30 pm.
- Extra Lecture Slot (to be used occasionally): Fri, 11 am – 12.30 pm.
- Tutorials: Mon, Tue, Wed 2 - 3 pm [for D3, D1, and D2 respectively].
- Tutors:
  - Prasant Singh (for D1 and second half of D3)
  - Avijit Panja (for D2 and first half of D3).
- Office Hours: Thu, 2 - 4 pm.
- For more information, see:
  - The "Booklet"
  - http://www.math.iitb.ac.in/~srg/courses/ma105-2016/

Textbook and Main References

- [TF] G. B. Thomas and R. L. Finney,*Calculus and Analytic Geometry*, 9th ed., Addison-Wesley/Narosa, 1998.
- [GL-1] S. R. Ghorpade and B. V. Limaye, *A Course in Calculus and Real Analysis* (ACICARA), Springer, 2006 (Fifth Indian Reprint, Springer (India), 2010). [For a dynamic errata, see: `http://www.math.iitb.ac.in/∼srg/acicara/`]
- [GL-2] S. R. Ghorpade and B. V. Limaye, *A Course in Multivariable Calculus and Analysis* (ACIMC), Springer, 2010 (First Indian Reprint, Springer (India), 2010). [For a dynamic errata, see: `http://www.math.iitb.ac.in/∼srg/acimc/`]
- Additional references are given in the Booklet.

Evaluation Plan

- Two Quizzes: each of 10% weightage and 40 minutes duration, on Sep 8 and Nov 10 at 2 pm.
- Mid-Sem: 30% weightage and 2 hours duration, on Sep 29 at 10 am.
- End-Sem: 50% weightage and 3 hours duration, on Nov 28 at 9.30 am.

Policy for Attendance

- Attendance is compulsory! [Also, attending lectures and tutorials will be good for you!]. Random attendance may be taken on random days. Those with less than 80 % attendance may get a DX grade.

Planned Absenses (for one week in each month)

- Week of 29 August–2 September. Make-up classes on Aug 12 and 19.
- Week of 19–23 September. Make-up classes on Sep 2 and 9.
- Week of 10–14 October. Make-up class on 30 Sep.
- Week of 31 October–4 November. Make-up class on 11 Nov.

NOTATIONS

$\mathbb{N}$ = the set of all positive integers

$= \{1, 2, 3, \ldots\}$

$\mathbb{Z}$ = the set of all integers

$= \{\ldots -2, -1, 0, 1, 2, 3, \ldots\}$

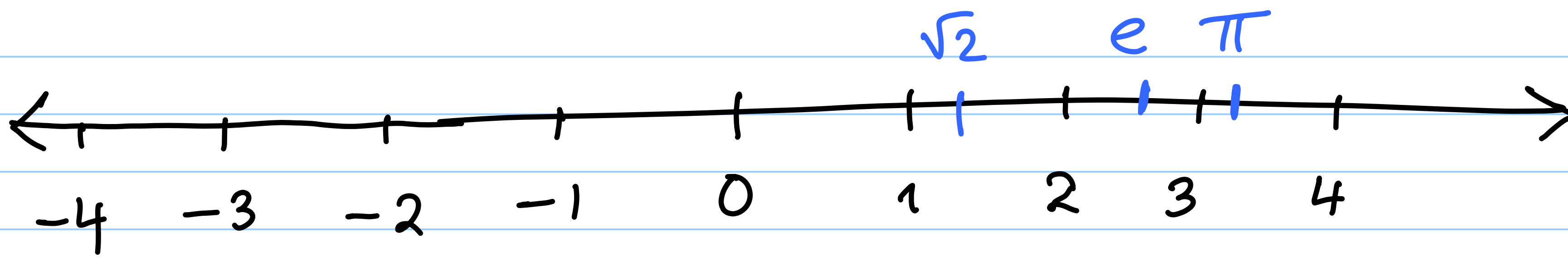
$\mathbb{Q}$ = the set of all rational numbers

$= \left\{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\right\}$

$\mathbb{R}$ = the set of all real numbers

$\mathbb{R}$ — rational numbers — algebraic numbers

$\mathbb{R}$ — irrational number — transcendental numbers



Proposition: $\sqrt{2}$ is irrational

Pf: Suppose not! Then $\sqrt{2} = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $q > 0$ and $GCD(p, q) = 1$.

So $2q^2 = p^2 \Rightarrow q \mid p^2 \Rightarrow q \mid p \rightarrow q = 1$

But there is no integer whose square is 2.

What we'll assume about the set $\mathbb{R}$

- **Algebraic Properties**
  - that we have operations of addition (+) and multiplication (·) satisfying the "usual" properties. [see, e.g., p.3 of GL-1]

- **Order Properties**
  - that there is a subset $\mathbb{R}^+$ of $\mathbb{R}$ s.t.
    i) Given $a \in \mathbb{R}$, exactly one of the following holds:
    $$a \in \mathbb{R}^+ \quad \text{OR} \quad a = 0 \quad \text{OR} \quad -a \in \mathbb{R}^+$$
    ii) $a, b \in \mathbb{R}^+ \implies a+b \in \mathbb{R}^+$ & $ab \in \mathbb{R}^+$.

Thanks to the existence of $\mathbb{R}^+$, we can <u>define</u> : $a < b$ if $b - a \in \mathbb{R}^+$.

(and $a > b$ if $b < a$). In particular, $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and

- $a, b \in \mathbb{R} \implies a < b$ or $a = b$ or $a > b$
- $a, b \in \mathbb{R}$ & $a < b \implies a + c < b + c \quad \forall c \in \mathbb{R}$,

$$ac < bc \quad \text{if } c > 0,$$

$$\& \ ac > bc \quad \text{if } c < 0.$$

Boundedness

Let S be a subset of $\mathbb{R}$.

Def: • S is bounded above if $\exists\, \alpha \in \mathbb{R}$ s.t. $x \leq \alpha \;\; \forall\, x \in S$.

Any such α is called an upper bound of S.

• S is bounded below if $\exists\, \beta \in \mathbb{R}$ s.t. $\beta \leq x \;\; \forall\, x \in S$

Any such β is called a lower bound of S.

• S is bounded if it is bounded above as well as below.

Supremum (lub) and Infimum (glb)

Let $S \subseteq \mathbb{R}$. A real number M is called a supremum or a least upper bound of S if

i) M is an upper bound of S, i.e.,

$$x \leq M \quad \forall x \in S.$$

ii) $M \leq \alpha$ for any upper bound α of S

Similarly, a real number m is called an infimum or a greatest lower bound of S if .....

Easy to see: If a supremum exists, then it is unique. We denote the supremum of a set S by $\sup S$.

Likewise if S has an infimum, then it is unique and is denoted by $\inf S$.

Example: ① $S = \{x \in \mathbb{R} : 0 < x \leq 1\}$.

$\inf S = 0$ $\qquad$ $\sup S = 1 = \max S$

② $S = \{x \in \mathbb{Q} : x^2 < 2\}$.

$\inf S = -\sqrt{2}$ $\qquad$ $\sup S = \sqrt{2}$.

Completeness Property (of $\mathbb{R}$):

Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum.

Consequences of the Completeness Property (together with algebraic & order properties):

1. Every nonempty subset of $\mathbb{R}$ that is bounded below has an infimum.

② [Archimedian Property]

Given $x \in \mathbb{R}$, $\exists\ n \in \mathbb{N}$ s.t. $n > x$.

③ [Existence of n^{th} roots]

Given $n \in \mathbb{N}$ and $a \in \mathbb{R}$ with $a \geq 0$, $\exists$ unique $b \in \mathbb{R}$ s.t. $b \geq 0$ and $b^n = a$.

[Denote this b by $a^{1/n}$ or by $\sqrt[n]{a}$.]

④ [Denseness of $\mathbb{Q}$ and of $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$]

Given any $a, b \in \mathbb{R}$ with $a < b$, there is a rational as well as an irrational between a & b.

Open, closed, and Semi-open Intervals

Given $a, b \in \mathbb{R}$, define :

$$(a,b) := \{x \in \mathbb{R} : a < x < b\}$$

$$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$

$$[a,b) :=$$

$$(a,b] :=$$

Also, it is useful to introduce symbols ∞ and $-\infty$ and define the

Semi-infinite and doubly infinite intervals:

$$(a, \infty) := \{x \in \mathbb{R} : x > a\}$$

$$[a, \infty) := \{x \in \mathbb{R} : x \geq a\}$$

$$(-\infty, b) := \{x \in \mathbb{R} : x < b\}$$

$$(-\infty, b] := \{x \in \mathbb{R} : x \leq b\}$$

and $(-\infty, \infty) := \mathbb{R}$.

<u>General Defn.</u>: A subset I of $\mathbb{R}$ is called an <u>interval</u> if $x, y \in I$, $x < y \Rightarrow [x, y] \subseteq I$.

Absolute Value

For $x \in \mathbb{R}$, $\quad |x| := \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$

Basic Facts: For any $x, y \in \mathbb{R}$,

- $|x+y| \leqslant |x| + |y|$ (Triangle inequality)
- $\big|\, |x| - |y| \,\big| \leqslant |x-y|$

Exercise: Show that for $a, b \in \mathbb{R}$ with $a \geqslant 0$, $b \geqslant 0$, and $n \in \mathbb{N}$, we have

$$\left| \sqrt[n]{a} - \sqrt[n]{b} \right| \leqslant \sqrt[n]{|a-b|} .$$