# MA 105 : Calculus

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# Lecture 11

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For any rational number $r$, we have

$$\frac{d}{dx}(x^r) = r x^{r-1} \quad \text{for all } x \in (0, \infty)$$

and hence by the Fundamental Theorem of Calculus,

$$\int_1^x t^s \, dt = \frac{x^{s+1}}{s+1} - \frac{1}{s+1} \quad \text{for any } x \in (0,\infty), \text{ provided } s \neq -1.$$

**Definition:** $\ln x := \int_1^x \frac{1}{t} \, dt$ for any $x \in (0, \infty)$.

(Natural Logarithm of $x$)

Note that this is well-defined since the function $f : (0,\infty) \to \mathbb{R}$ defined by $f(t) = \frac{1}{t}$ is continuous, and hence it is Riemann-integrable on every closed & bounded subinterval of $(0,\infty)$.

## Basic Properties of the logarithmic function ln :

① $\ln : (0, \infty) \to \mathbb{R}$ is differentiable and

$$(\ln)'(x) = \frac{1}{x} \quad \forall\ x \in (0, \infty)$$

② $\ln : (0, \infty) \to \mathbb{R}$ is strictly increasing & strictly concave

③ $\ln(x_1 x_2) = \ln x_1 + \ln x_2 \quad \forall\ x_1, x_2 \in (0, \infty)$

④ $\ln x \to \infty$ as $x \to \infty$ and $\ln x \to -\infty$ as $x \to 0^+$.

⑤ $\ln ; (0, \infty) \to \mathbb{R}$ is one-one and onto.

**Proof:** ① By the FTC and the continuity of $t \mapsto \frac{1}{t}$, $\ln$ is diff. on $(0,\infty)$ and $(\ln)'(x) = \frac{1}{x}$.

② $(\ln)'(x) = \frac{1}{x} > 0 \ \forall x \in (0,\infty) \Rightarrow \ln$ is strictly increasing

and

$(\ln)''(x) = \frac{-1}{x^2} < 0 \ \forall x \in (0,\infty) \Rightarrow \ln$ is strictly concave.

③ Let $x_1, x_2 \in (0,\infty)$. Consider $h : (0,\infty) \to \mathbb{R}$ defined by

$$h(x) = \ln(x x_2) - \ln x \qquad \text{for } x \in (0,\infty).$$

Then $h$ is differentiable and

$$h'(x) = \frac{1}{x x_2} \cdot x_2 - \frac{1}{x} = 0 \qquad \forall x \in (0,\infty).$$

Hence $h$ is a constant fn $\Rightarrow h(x) = h(1) = \ln x_2 - \ln 1$
$= \ln x_2$.

This yields $\ln x_1 x_2 = \ln x_1 + \ln x_2$.

(4) By (3) above

$$\ln 2^n = n \ln 2 \longrightarrow \infty \quad \text{as } n \to \infty \qquad (\because \ln 2 > 0)$$

and since $\ln$ is strictly increasing on $(0,\infty)$, we see that

$$\ln x \to \infty \quad \text{as } x \to \infty.$$

Likewise

$$\ln 2^{-n} = -n \ln 2 \to -\infty \quad \text{as } n \to \infty$$

and hence

$$\ln x \to -\infty \quad \text{as } x \to 0^+.$$

(5) $\ln$ is one-one since it is strictly increasing. Also given any $y \in \mathbb{R}$, $\exists\ x_1, x_2 \in (0,\infty)$ s.t. $\ln x_1 < y < \ln x_2$, by (4) above. Hence by the IVP, $\exists\ x \in (x_1, x_2) \subseteq (0,\infty)$ such that $\ln x = y$. So $\ln : (0,\infty) \to \mathbb{R}$ is onto.

**Corollary**: $\displaystyle\sum_{k=1}^{n} \frac{1}{k} \to \infty$ as $n \to \infty$.

**Pf**: $\displaystyle \ln n = \int_1^n \frac{1}{t}\,dt = \sum_{k=2}^{n} \int_{k-1}^{k} \frac{1}{t}\,dt \leq \sum_{k=2}^{n} \frac{1}{k-1}$.

In other words

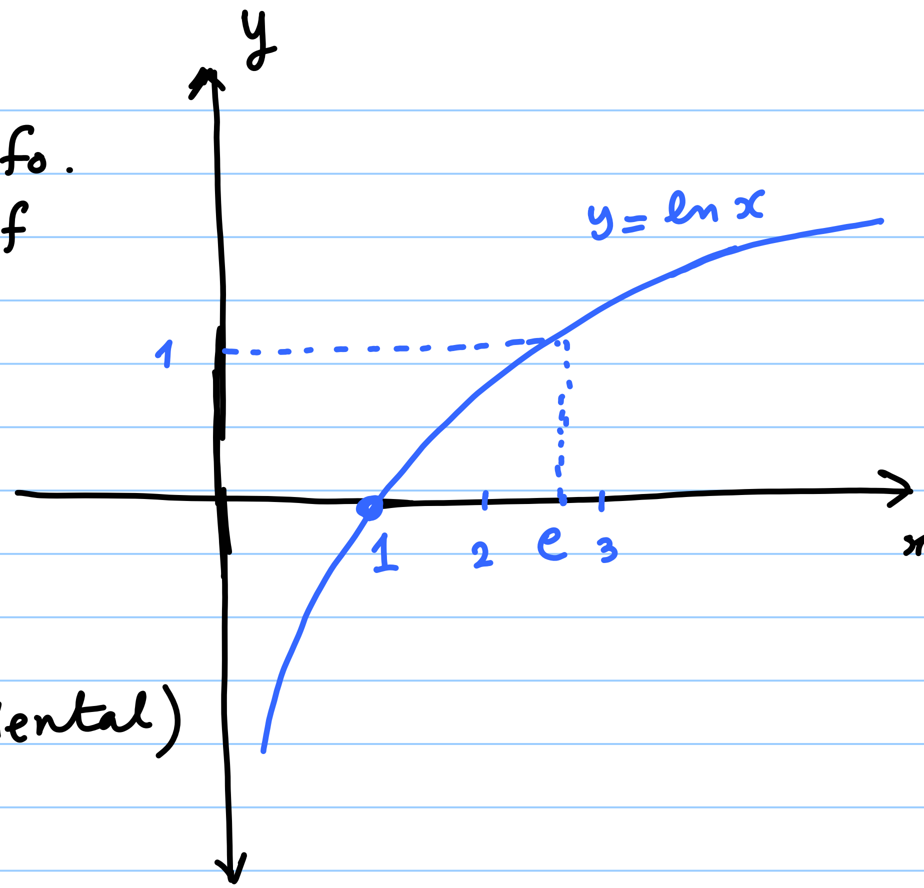
$$\sum_{k=1}^{n} \frac{1}{k} \geq \ln(n+1) \to \infty \text{ as } n \to \infty \text{ since } \ln x \to \infty \text{ as } x \to \infty.$$

Note also that $\ln 2 \leq 1$, while

$$\ln 4 = \int_1^4 \frac{1}{t}\,dt = \int_1^2 \frac{1}{t}\,dt + \int_2^4 \frac{1}{t}\,dt \geq \frac{1}{2} + \frac{1}{4}\cdot 2 = 1.$$

Thus $\ln 2 \leq 1 \leq \ln 4$ and so by IVP, $\exists$ unique $e \in \mathbb{R}$ such that $2 \leq e \leq 4$ and $\ln e = 1$.

One can use the above info. to make a rough sketch of the graph of ln :




Remark: One has

$e = 2.71828182\ldots$

and it can be shown that $e$ is irrational (and in fact $e$ is transcendental)

# Exponential Function

Since $\ln : (0,\infty) \longrightarrow \mathbb{R}$ is 1-1 & onto, it has an inverse, which is denoted by exp. Note that

$$\exp : \mathbb{R} \longrightarrow (0,\infty)$$

is the function determined by the relation

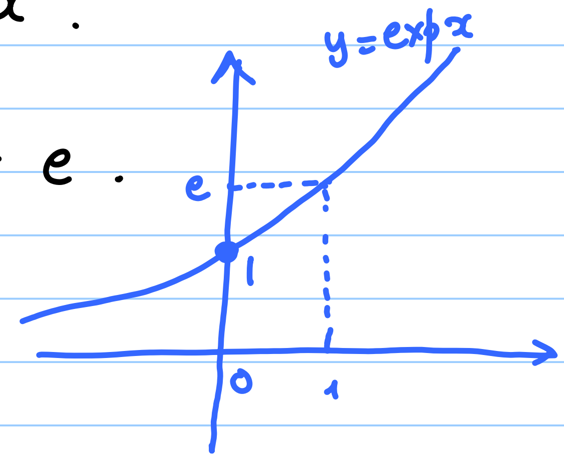
$$\exp x = y \iff \ln y = x .$$

In particular,

$$\exp 0 = 1 \quad \text{and} \quad \exp 1 = e .$$

and of course

$$\exp x > 0 \quad \forall x \in \mathbb{R} .$$



# Basic Properties of the exponential function

① $\exp : \mathbb{R} \to (0, \infty)$ is differentiable and

$$(\exp)'(x) = \exp(x) \quad \forall\, x \in \mathbb{R}$$

② $\exp : \mathbb{R} \to (0, \infty)$ is strictly increasing & strictly convex on $\mathbb{R}$.

③ $\exp(x_1 + x_2) = (\exp x_1)(\exp x_2) \quad \forall\, x_1, x_2 \in \mathbb{R}$

④ $\exp x \to \infty$ as $x \to \infty$ and $\exp x \to 0$ as $x \to -\infty$.

Proof: ① if $x = \ln y$ then $(\exp)'(x) = \dfrac{1}{(\ln)'(y)} = y = \exp x$,

thanks to the Differentiable inverse theorem.

② Follows from ①.

(3) $\exp(x_1) = y_1$ , $\exp(x_2) = y_2$ and

$$\ln(y_1 y_2) = \ln y_1 + \ln y_2 = x_1 + x_2$$

$$\Rightarrow \exp(x_1 + x_2) = y_1 y_2 = \exp(x_1)\exp(x_2).$$

(4) range $(\exp) = (0, \infty)$ and $\exp$ is strictly increasing on $\mathbb{R}$

$$\Rightarrow \exp x \to \infty \text{ as } x \to \infty \text{ \& } \exp x \to 0 \text{ as } x \to -\infty.$$

Note: $(\exp)^{(n)}(x) = \exp x \quad \forall n \geq 1$ & in particular $(\exp)^{(n)}(0) = 1$.

So the $n^{th}$ Taylor polynomial of exp around 0 is

$$P_n(x) = \exp 0 + \sum_{k=1}^{n} \frac{\exp^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

# Real Powers of positive numbers

First note that for any $r \in \mathbb{Q}$ and $a \in (0, \infty)$

$$\ln a^r = r \ln a$$

[e.g. since the derivative of $\ln x^r - r \ln x$ is $0$ and since its value at $x=1$ is also $0$.]

Defn: For any $b \in \mathbb{R}$ and $a \in (0,\infty)$

$$\boxed{a^b := \exp(b \ln a)}$$

$a \to$ base

$b \to$ exponent

or equivalently,

$$\ln a^b = b \ln a.$$

Note that this defn. is consistent with the defn. of rational powers.

Basic Properties : For all $a, a_1, a_2 \in (0,\infty)$ and $b, b_1, b_2 \in \mathbb{R}$

- $a^{b_1+b_2} = a^{b_1} a^{b_2}$
- $(a_1 a_2)^b = a_1^b a_2^b$
- $(a^{b_1})^{b_2} = a^{b_1 b_2}$
- For fixed $a \in (0,\infty)$, the function $f_a : \mathbb{R} \to (0,\infty)$ defined by $f_a(x) = a^x$ is differentiable and $f_a'(x) = a^x (\ln a)$.
- For fixed $b \in \mathbb{R}$, the function $g_b : (0,\infty) \to (0,\infty)$ defined by $g_b(x) = x^b$ is differentiable and $g_b'(x) = b x^{b-1}$.

**Proof:** Everything follows easily from the defn:

$$a^b := \exp(b \ln a)$$

and the properties of ln and exp proved earlier.

**Remark:** Taking $a = e$, we find

$$e^b = \exp(b \ln e) = \exp b, \quad \forall b \in \mathbb{R}$$

With this in view, we may write

$$\exp x = e^x$$

and in particular,

$$a^b = e^{b \ln a}.$$

# Applications of Riemann Integration

We have seen already that the notion of Riemann integration can be used to define and determine the AREA of a large class of planar regions, e.g.

- Area between two curves:

If $R = \{(x, y) : a \leq x \leq b \;\&\; f_1(x) \leq y \leq f_2(x)\}$ where $f_1, f_2 : [a, b] \to \mathbb{R}$ are Riemann integrable functions such that $f_1 \leq f_2$ i.e., $f_1(x) \leq f_2(x)$, $\forall x \in [a, b]$, then

$$\text{Area}(R) = \int_a^b [f_2(x) - f_1(x)]\, dx .$$

Similarly if $R = \{(x, y) : c \leq y \leq d \;\&\; g_1(y) \leq x \leq g_2(y)\}$ where $g_1, g_2 : [c, d] \to \mathbb{R}$ are integrable and $g_1 \leq g_2$, then

$$\text{Area}(R) = \int_c^d [g_2(y) - g_1(y)]\, dy .$$

<u>Examples</u>: ① Find the area of the region bounded by the parabolas $x = -2y^2$ and $x = 1-3y^2$.

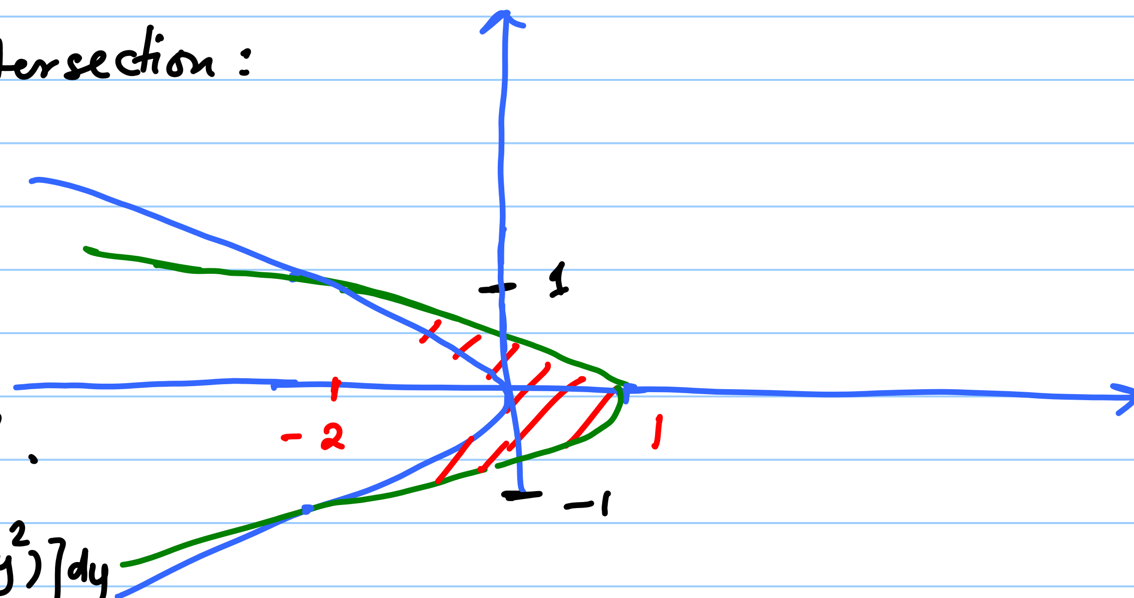
First find points of intersection:

$$-2y^2 = 1-3y^2$$

$$\Longrightarrow y = \pm 1$$

Note that

$$1-3y^2 \geq -2y^2 \quad \forall y \in [-1,1].$$




So

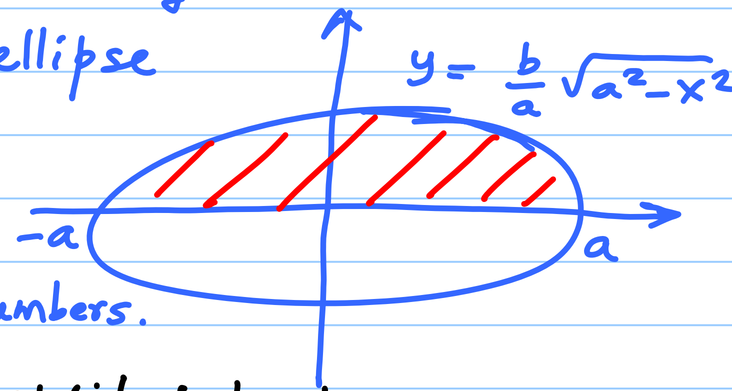
$$\text{Area} = \int_{-1}^{1} \left[(1-3y^2) - (-2y^2)\right] dy$$

$$= \int_{-1}^{1} (1-y^2)\, dy = \frac{4}{3}.$$

② Consider the problem of finding the area of the region enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a, b$ are positive real numbers.




The given region can be split into two non-overlapping region and hence the desired area is

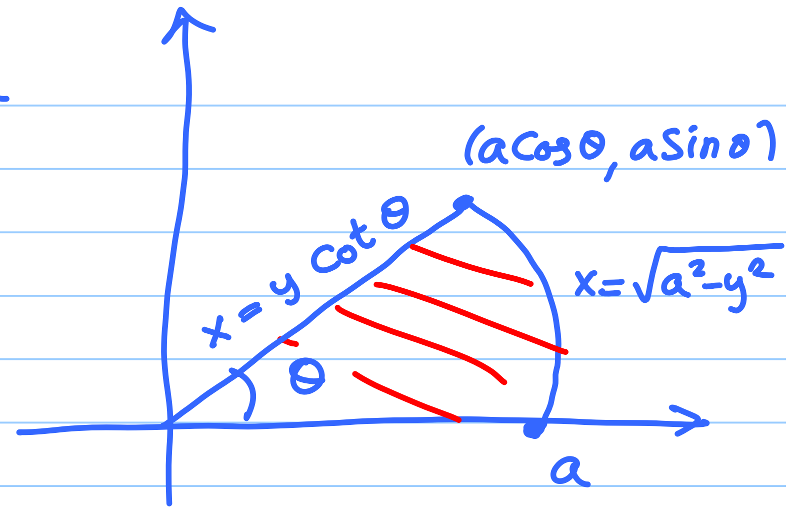
$2 \times$ Area ( region between $y = \frac{b}{a}\sqrt{a^2-x^2}$, $x \in [-a,a]$ & $y=0$)

$$= 2\int_{-a}^{a} \frac{b}{a}\sqrt{a^2-x^2} = \frac{2b}{a}\int_{-\pi/2}^{\pi/2} a^2\cos^2\theta \, d\theta = \pi ab.$$

As a special case (viz. $b=a$) we find that

Area of a disk of radius $a$ $= \pi a^2$.

(3) Consider the sector of a disk of radius a, which subtends an angle $\theta$ at the center.




This can be viewed as the region bounded by curves

$$x = y\cot\theta \quad \& \quad x = \sqrt{a^2-y^2}, \qquad 0 \le y \le a\sin\theta.$$

Hence the desired area is

$$\int_0^{a\sin\theta} \left[\sqrt{a^2-y^2} - (\cot\theta)\,y\right] dy = a^2\int_0^{\theta} \cos^2 t\,dt - (\cot\theta)\frac{a^2\sin^2\theta}{2}$$

$$= \frac{a^2\theta}{2}.$$

## Review of polar coordinates

For any point $P = (x, y)$ of the plane $\mathbb{R}^2$, numbers $r$ and $\theta$ satisfying

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

are called polar coordinates of P. More precisely and for definitiveness, note that if P is not the origin, i.e., if $(x, y) \neq (0, 0)$, then there are unique $r, \theta \in \mathbb{R}$ such that

$$\boxed{r > 0, \quad \theta \in (-\pi, \pi], \quad x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.}$$

We refer to the pair $(r, \theta)$ as <u>the</u> polar coordinates of P. Note that $r = \sqrt{x^2+y^2}$. <u>Exer</u>: Express $\theta$ as a fn. of $x, y$.

## Polar Curves or Curves given by a polar equation

An equation in the polar coordinates $r$ and $\theta$, such as an equation of the form

$$r = p(\theta) \quad , \quad \theta \in [\alpha, \beta]$$

(where usually $\alpha, \beta \in [-\pi, \pi]$) determines a curve in the plane, namely

$$\{ (p(\theta)\cos\theta, \ p(\theta)\sin\theta) \ : \ \theta \in [\alpha, \beta] \}$$

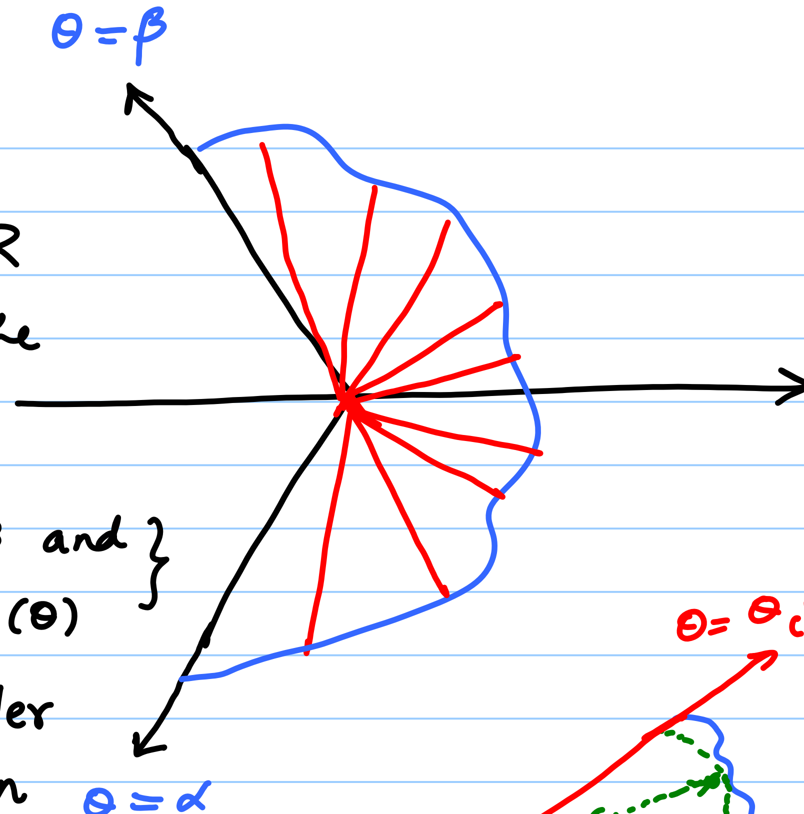
Remark: Several examples and pictures of such curves can be found in pages 262-263 of [GL-1].

Now consider the planar region bounded by a polar curve $r = p(\theta)$ and the rays $\theta = \alpha$ & $\theta = \beta$.

Assume that the function

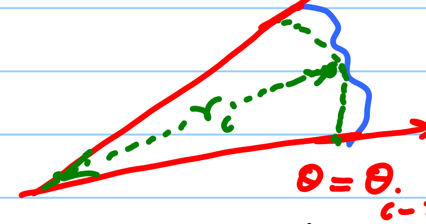
$$p : [\alpha, \beta] \longrightarrow \mathbb{R}$$

is integrable. The region $R$ enclosed by $r = p(\theta)$ and by the rays $\theta = \alpha$ and $\theta = \beta$, i.e.,




$$R = \left\{ (r\cos\theta, r\sin\theta) \ : \ \begin{array}{l} \alpha \le \theta \le \beta \text{ and} \\ 0 \le r \le p(\theta) \end{array} \right\}$$

can be subdivided into smaller regions by taking a partition




$\{\theta_0, \theta_1, \theta_2, \ldots, \theta_n\}$ of $[\alpha, \beta]$.

The area of the subregion between $\theta = \theta_{i-1}$ and $\theta = \theta_i$ can be approximated by the area of a sector of disk of radius $r_i$

where $r_i = p(s_i)$, $s_i \in [\theta_{i-1}, \theta_i]$. In view of Example 3 above, this area is equal to

$$\frac{1}{2} r_i^2 \theta_i - \frac{1}{2} r_i^2 \theta_{i-1} = \frac{1}{2} r_i^2 (\theta_i - \theta_{i-1})$$

and hence the area of R is approximately

$$\sum_{i=1}^{n} \frac{1}{2} r_i^2 (\theta_i - \theta_{i-1}) = \sum_{i=1}^{n} \frac{1}{2} p(s_i)^2 (\theta_i - \theta_{i-1}).$$

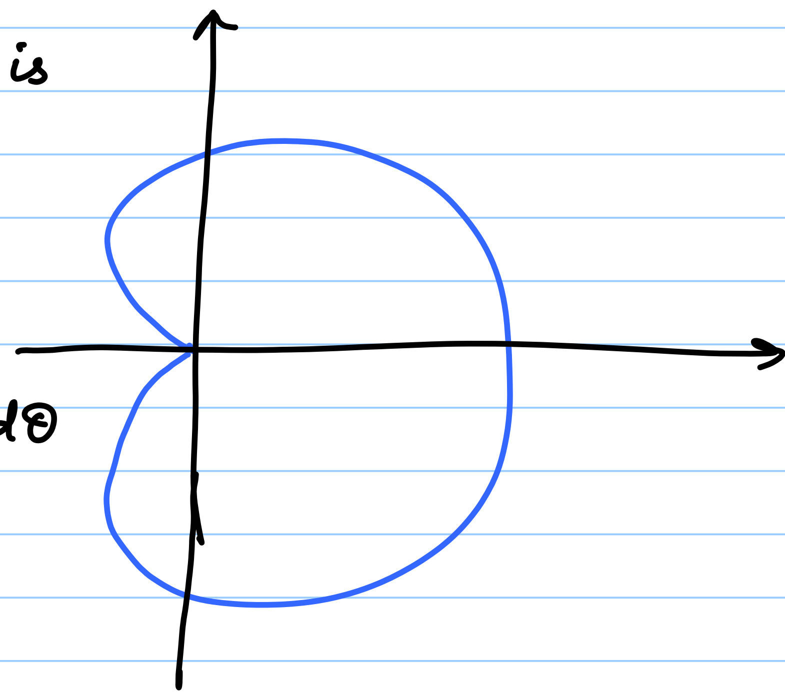
It seems clear that these approximations become better as the partition $\{\theta_0, \theta_1, \ldots, \theta_n\}$ gets finer and finer. With this in view we define

$$\text{Area}(R) = \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta = \frac{1}{2} \int_{\alpha}^{\beta} p(\theta)^2 \, d\theta .$$

**Example** Consider the cardioid

$$r = a(1+\cos\theta) \qquad \theta \in (-\pi, \pi].$$

The area enclosed by this is



$$\frac{1}{2}\int_{-\pi}^{\pi} [a(1+\cos\theta)]^2 \, d\theta$$

$$= \frac{a^2}{2}\int_{-\pi}^{\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{3a^2\pi}{2}.$$