MA 105 : Calculus

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Arc Length of a Curve:

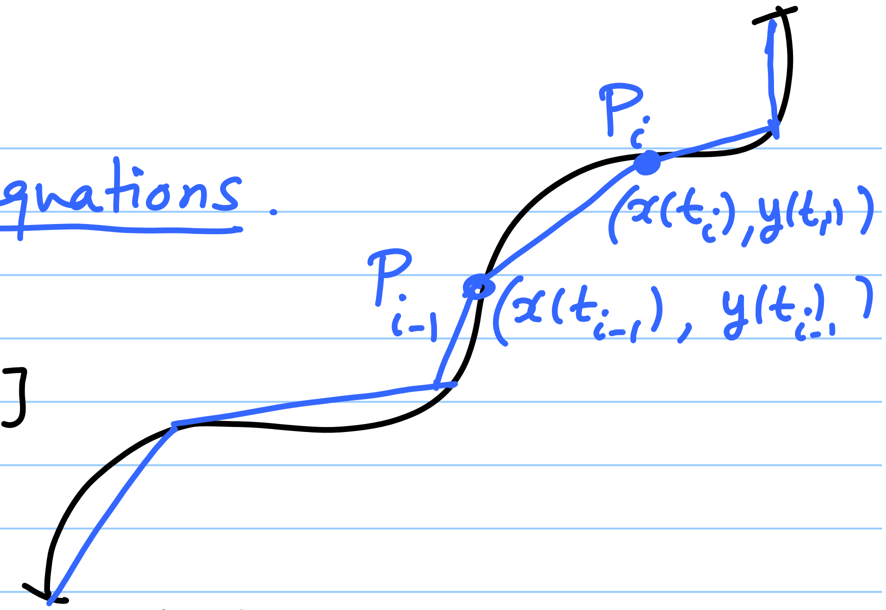
Consider the problem of determining the "length" of a curve. Our basic assumption will be that

length of a line segment joining points (x_1, y_1) & (x_2, y_2) $= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

To pass on to an arbitrary curve, we'll use ideas similar to those used in arriving at a suitable notion of area via Riemann integration.

But first let us make it clear what we shall mean by a "curve". In short we mean a

Parametrized curve or a curve given by parametric equations.



Such a curve is given by

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}, \quad t \in [\alpha, \beta]$$

where $x, y : [\alpha, \beta] \to \mathbb{R}$ are functions. We'll assume that this curve, say C, is smooth, which means that x, y are differentiable functions whose derivatives x', y' are continuous.

Now take a partition $\{t_0, t_1, \dots, t_n\}$ of the parameter interval $[\alpha, \beta]$. The curve C could be approximated by line segments joining P_0 to P_1, P_1 to P_2, $\dots$, P_{n-1} to P_n where $P_i = (x(t_i), y(t_i))$.

Hence

$$\ell(C) \sim \sum_{i=1}^{n} \text{distance from } P_{i-1} \text{ to } P_i$$

$$= \sum_{i=1}^{n} \sqrt{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2}$$

$$= \sum_{i=1}^{n} \left(\sqrt{x'(c_i)^2 + y'(c_i^*)^2} \right)(t_i - t_{i-1}).$$

With this in view, we define

the arc length of $C = \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2}\, dt$

Special Cases :

Curves of the form $y = f(x)$, $x \in [a,b]$.

Here a parametrization is

$$C : \begin{cases} x = t \\ y = f(t) \end{cases} \qquad t \in [a,b]$$

and if f is continuously differentiable, then we find

$$\ell(C) = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Similarly for curves given by $x = g(y)$, $y \in [c,d]$,

$$\ell(C) = \int_c^d \sqrt{1 + g'(y)^2}\, dy.$$

Example (Tut Sheet No. 5, Q. 4)

Find the arc-length of

i) cycloid : $x = t - \sin t, \quad y = 1 - \cos t, \quad 0 \le t \le 2\pi$

ii) curve $y = \int_0^x \sqrt{\cos 2t}\, dt \quad , 0 \le x \le \frac{\pi}{4}$

Solution:

i) $$l(C) = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt$$

$$= \int_0^{2\pi} 2 \left| \sin \frac{t}{2} \right| dt = 8$$

ii) $$l(C) = \int_0^{\pi/4} \sqrt{1+(y')^2}\, dx = \int_0^{\pi/4} \sqrt{1+\cos 2x}\, dx$$

$$= \sqrt{2} \int_0^{\pi/4} \cos x\, dx = 1.$$

Example (Tut Sheet No. 5, Q. 5 - pruned version)

Find the arc length of

$$y = \frac{x^3}{3} + \frac{1}{4x}, \quad 1 \le x \le 3.$$

Solution: We have

$$y' = x^2 - \frac{1}{4x^2} \implies 1 + (y')^2 = 1 + x^4 - \frac{1}{2} + \frac{1}{16x^4}$$

$$= \left(x^2 + \frac{1}{4x^2}\right)^2$$

and hence

$$\ell(C) = \int_1^3 \left(x^2 + \frac{1}{4x^2}\right) dx = \frac{53}{6}.$$

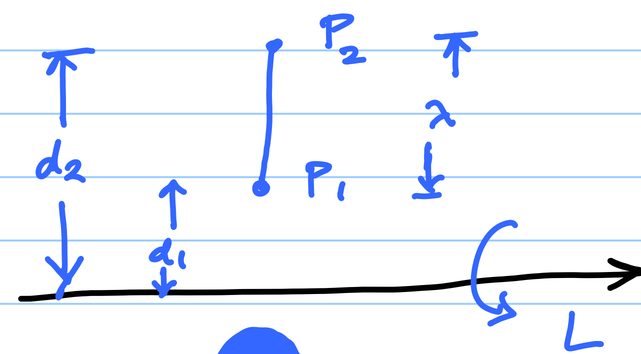
Surface Area of Surfaces of Revolution

We will discuss here the notion of area for a surface of revolution. Such a surface is generated when a curve is revolved about a line. The basic idea in arriving at a suitable definition and formula for the same area is essentially the same as before:

analyze the surface obtained by revolving a small line segment about a line L and figure out what the area of the surface of revolution thus obtained ought to be. Then in the case of a general curve revolved about L, partition the parameter interval so as to reduce to the above case on each piece; sum up and pass to a limit so as to arrive at a suitable Riemann integral.

Suppose C is the line segment P_1P_2 with endpoints $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, and this is revolved about a line L that does not cross P_1P_2. Let d_1, d_2 denote the distance of P_1, P_2 from the line L and let λ denote the length of the line segment P_1P_2.

We distinguish three cases.



Case 1: $P_1P_2 \perp L$

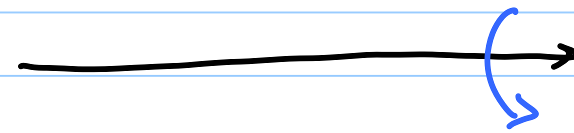
The surface of revolution is a washer and its surface area is

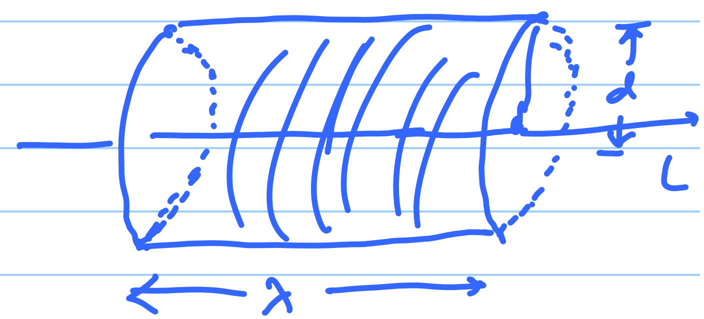
$$\left|\pi d_2^2 - \pi d_1^2\right| = \pi\,|d_2 - d_1|\,(d_2 + d_1)$$

$$= \pi(d_1 + d_2)\lambda .$$

Case 2 : $P_1 P_2 \parallel L$

In this case $d_1 = d_2 = d$ (say) and the surface of revolution is a right circular cylinder of radius d and length λ. To find its surface area, one may slit open this cylinder along a straight line parallel to L so as to obtain a rectangle of sides $2\pi d$ and λ. Hence

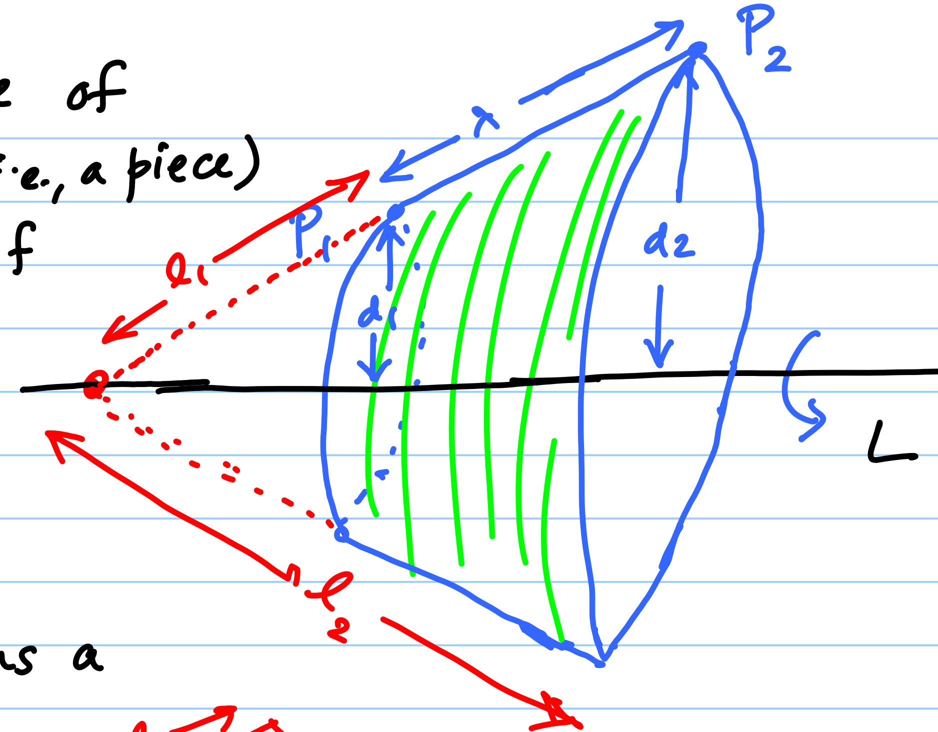




$$\text{Surface area} = 2\pi d \lambda = \pi (d_1 + d_2) \lambda .$$

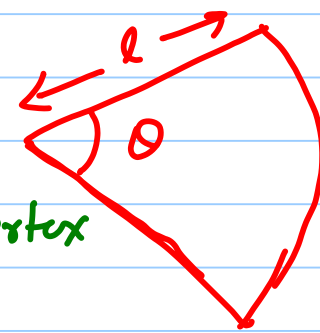
Case 3: $P_1P_2 \not\perp L$ and $P_1P_2 \not\parallel L$.

In this case the surface of revolution is a frustum (i.e., a piece) of a right circular cone of base radii d_1 and d_2.



To find the surface area, let us first note that the area of a right circular cone of base radius a and slant height ℓ is

$A = \pi \ell a$



$\ell\theta = 2\pi a$

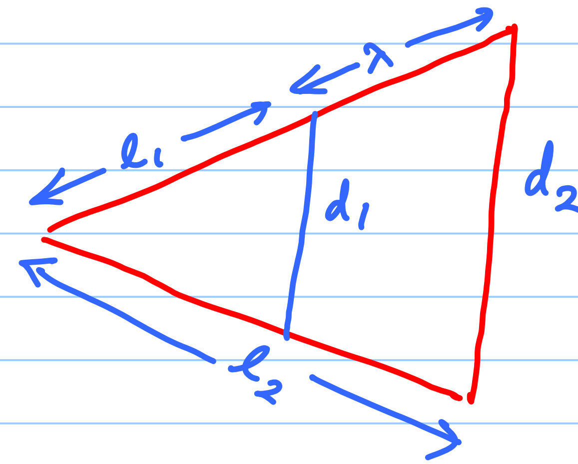
[To see this, slit open the cone along a line from its vertex to a point in its base. This gives a sector of a circle of radius ℓ s.t. length of its arc is $2\pi a$. Hence $\ell\theta = 2\pi a \Rightarrow \theta = 2\pi a/\ell$. So $A = \frac{1}{2}\ell^2\theta = \pi \ell a$.]

Going back to the frustum of the cone when T_1P_2 is revolved about L, we see from the above that

$$\text{surface area} = \pi d_2 \ell_2 - \pi d_1 \ell_1$$

assuming, without loss of generality, that $d_1 < d_2$

Using similar triangles,



$$d_1 \ell_2 = d_2 \ell_1$$

and hence

$$\pi d_2 \ell_2 - \pi d_1 \ell_1$$
$$= \pi (d_2 \ell_2 - d_2 \ell_1 + d_1 \ell_2 - d_1 \ell_1)$$
$$= \pi (d_1 + d_2)(\ell_2 - \ell_1)$$
$$= \pi (d_1 + d_2) \lambda .$$

Thus in either case the area of the surface of revolution, when P_1P_2 is revolved about L, is

$$\pi(d_1 + d_2)\lambda$$

where

$$d_i = \text{distance of } P_i \text{ from } L \quad \text{for } i = 1, 2$$

and

$$\lambda = \text{length}(P_1P_2).$$

In general, if C is a parametrically defined curve given by $(x(t), y(t))$, $t \in [\alpha, \beta]$ and if

$$\{t_0, t_1, \ldots, t_n\} \text{ is a partition of } [\alpha, \beta]$$

and if $P_i = (x(t_i), y(t_i))$, then the area of the surface of revolution generated by revolving C

about the line L can be approximated by

$$\sum_{i=1}^{n} \text{Area}\,(\text{frustum of cone generated by } P_{i-1} P_i)$$

$$= \sum_{i=1}^{n} \pi \,(d_{i-1} + d_i)\,\lambda_i$$

where

$$d_i = \text{dist.}(P_i, L) = \frac{|a\,x(t_i) + b\,y(t_i) + c|}{\sqrt{a^2+b^2}}$$

assuming that the line L is given by $ax+by+c=0$

and

$$\lambda_i = \text{length}\,(P_{i-1} P_i) = \sqrt{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2}$$

Note that if x, y are smooth, then by the MVT,

$$\lambda_i = \sqrt{x'(c_i)^2 + y'(c_i^*)^2}\,(t_i - t_{i-1}) \quad \forall i$$

Thus, as the partition $\{t_0, t_1, \ldots, t_n\}$ becomes finer and finer, both $\sum_{i=1}^{n} d_{i-1}\lambda_i$ and $\sum_{i=1}^{n} d_i \lambda_i$ seem to approach

$$\int_\alpha^\beta \frac{|ax(t)+by(t)+c|}{\sqrt{a^2+b^2}} \sqrt{x'(t)^2+y'(t)^2}\,dt$$

With this in view, we DEFINE

the area of the surface of revolution generated by revolving C about L $= 2\pi \int_\alpha^\beta \rho(t)\sqrt{x'(t)^2+y'(t)^2}\,dt$

where $\rho(t) := \text{dist.}\left((x(t), y(t)), L\right) = \dfrac{|ax(t)+by(t)+c|}{\sqrt{a^2+b^2}}$

Note that it is assumed that C doesn't cross L, i.e.,

$ax(t)+by(t)+c \geq 0 \quad \forall t$ or $ax(t)+by(t)+c \leq 0$
$\forall t \in [\alpha, \beta]$.

<u>Special Cases</u>:

1. L is the x-axis and C is given by $y=f(x)$, $x \in [a,b]$ where f is continuously differentiable and $f \geq 0$ or $f \leq 0$. For the corresponding surface of revolution, say S, we have

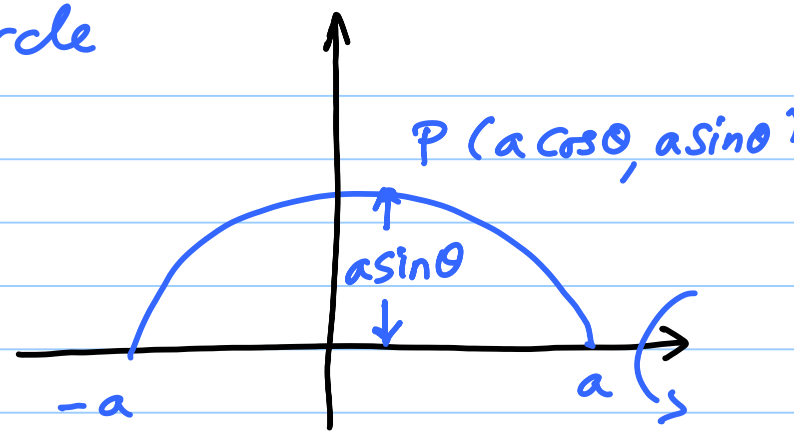
$$\text{Area}(S) = 2\pi \int_a^b |f(x)| \sqrt{1+f'(x)^2}\, dx.$$

2. L is the y-axis and C is given by $x=g(y)$, $y \in [c,d]$. As before,

$$\text{Area}(S) = 2\pi \int_c^d |g(y)| \sqrt{1+g'(y)^2}\, dy.$$

<u>Example</u> : Consider the semicircle

$$C : \begin{cases} x = a\cos\theta \\ y = a\sin\theta \end{cases}, \ \theta \in [0, \pi]$$



revolved about the x-axis.

The surface thus obtained is a sphere of radius a, and its surface area is

$$2\pi \int_0^{\pi} (a\sin\theta)\sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\, d\theta$$

$$= 2\pi a^2 \int_0^{\pi} \sin\theta \, d\theta$$

$$= 4\pi a^2.$$

Example (Tut sheet No. 5, Q. 5 - full version)

For the following curve, find the arc-length as well as the area of the surface obtained by revolving it about the line $y = -1$.

$$y = \frac{x^3}{3} + \frac{1}{4x}, \qquad 1 \leq x \leq 3.$$

We have seen earlier that for the given curve C,

$$\ell(C) = \int_1^3 \sqrt{1 + (y')^2}\, dx = \int_1^3 \left(x^2 + \frac{1}{4x^2}\right) dx = \frac{53}{6}$$

For the surface S obtained by revolving C about $y = -1$,

$$\text{Area}(S) = 2\pi \int_1^3 (y+1)\sqrt{1+(y')^2}\, dx = 2\pi \int_1^3 \left(\frac{x^3}{3} + \frac{1}{4x} + 1\right)\left(x^2 + \frac{1}{4x^2}\right) dx$$

$$= 1823\pi/18.$$

Note Title 9/19/2011

We now begin the theory of

Functions of Several Variables

also known as

Multivariable Calculus

Our basic references will be :

[TF] Thomas & Finney — Calculus and Analytic Geometry, 9th Ed., 1998.

[GL-2] Ghorpade & Limaye — A Course in Multivariable Calculus & Analysis, 2010

http://www.math.iitb.ac.in/~srg/acimc/