MA 105 : Calculus

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In the previous lecture, we began discussing the theory of functions of several variables, i.e.

$$f : D \to \mathbb{R} \quad \text{where } D \subseteq \mathbb{R}^n$$

and have thus far discussed the notions of

- Sequences in $\mathbb{R}^2$ and their convergence
- Limits of functions of two variables
- Continuity of functions of two variables

Recall:

For $D \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in D$, a function $f : D \to \mathbb{R}$ is said to be <u>continuous</u> at (x_0, y_0) if $\{(x_n, y_n)\}$ seq. in D & $(x_n, y_n) \to (x_0, y_0) \implies f(x_n, y_n) \to f(x_0, y_0)$.

Equivalently,

for every $\varepsilon > 0$, $\exists\, \delta > 0$ such that
$$(x,y) \in S_\delta(x_0,y_0) \cap D \implies |f(x,y) - f(x_0,y_0)| < \varepsilon.$$

and one can also replace the open square
$$S_\delta(x_0,y_0) := \{(x,y) \in \mathbb{R}^2 : |x-x_0| < \delta \text{ and } |y-y_0| < \delta\}$$
by the open disk
$$\mathbb{B}_\delta(x_0,y_0) := \{(x,y) \in \mathbb{R}^2 : \underbrace{|(x,y)-(x_0,y_0)| < \delta}_{\text{i.e., } \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta}\}$$

As in the case of functions of one variable, this is a "local notion". Globally, we say that

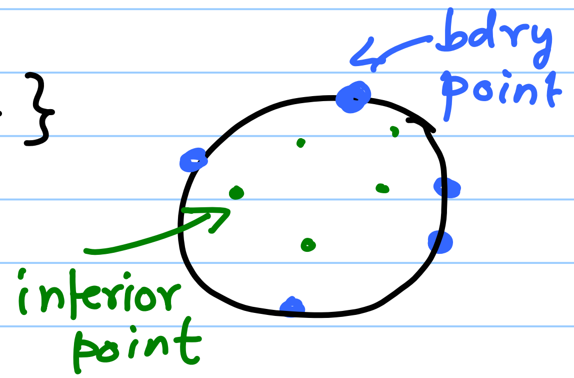
$f : D \to \mathbb{R}$ is <u>continuous</u> (on D) if it is continuous at every $(x_0,y_0) \in D$.

To state analogues of properties of continuous functions of one variable such as the IVP, we need the following definitions.

Let $D \subseteq \mathbb{R}^2$. We say that a point $(x_0, y_0) \in \mathbb{R}^2$ is (i) an <u>interior point</u> of D if $S_r(x_0, y_0) \subseteq D$ for some $r > 0$

(ii) a <u>boundary point</u> of D if for every $r > 0$, $S_r(x_0, y_0)$ contains a point of D as well as a point not in D.

<u>Example</u>: If $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is the closed unit disk, then the points of $\mathbb{B}_1(0,0)$ are the interior points while those on the circle $x^2 + y^2 = 1$ are the boundary points.



More definitions: A subset D of $\mathbb{R}^2$ is said to be

- **closed** if every boundary point of D is in D.
- **open** if every point of D is its interior point.
- **bounded** if $D \subseteq S_r(0,0)$ for some $r > 0$.
- **path-connected** if for any two points (x_0, y_0) and (x_1, y_1) of D, $\exists$ continuous functions

  $$x, y : [\alpha, \beta] \longrightarrow \mathbb{R}$$

  such that

  $$(x(\alpha), y(\alpha)) = (x_0, y_0), \quad (x(\beta), y(\beta)) = (x_1, y_1)$$

  and $$(x(t), y(t)) \in D \quad \forall\, t \in [\alpha, \beta].$$

(i.e., any two points of D can be joined by a path in D)

We can now state the analogue of IVP etc.

Theorem: Let $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be continuous

(1) If D is path-connected, then f has the IVP on I, i.e., if r is any real number between $f(P_1)$ and $f(P_2)$ for some $P_1, P_2 \in D$, then

$$r = f(P) \quad \text{for some } P \in D.$$

[Consequently, $f(D)$ is an interval in $\mathbb{R}$.]

(2) If D is closed and bounded, then f is a bounded function (i.e., $\exists\, M \in \mathbb{R}$ such that $|f(x,y)| \leq M \ \forall\, (x,y) \in D$) and further, f attains its bounds, i.e. $\exists\, P_1, P_2 \in D$ such that

$f(P_1) = \inf \{ f(x,y) : (x,y) \in D \}$ and

$f(P_2) = \sup \{ f(x,y) : (x,y) \in D \}$

If, in addition, D is path-connected, then the range of f is the interval $[f(P_1), f(P_2)]$.

Partial Differentiation

Let $D \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in D$ be an interior point of D. Let $f : D \to \mathbb{R}$ be a function.

Defn: The partial derivative of f with respect to x at the point (x_0, y_0) is the limit

$$\lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

provided this limit exists. It is denoted by $f_x(x_0, y_0)$.

Likewise, the partial derivative of f w.r.t. y at (x_0, y_0) is the limit

$$\lim_{k \to 0} \frac{f(x_0, y_0+k) - f(x_0, y_0)}{k}$$

provided this limit exists. It is denoted by $f_y(x_0, y_0)$

Other notations for $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$:

$$\frac{\partial f}{\partial x}(x_0, y_0) \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0)$$

<u>Defn</u>: If both the partial derivatives of f at (x_0, y_0) exist, then the pair

$$\nabla f(x_0, y_0) = \left(f_x(x_0, y_0), \ f_y(x_0, y_0) \right)$$

is called the <u>gradient</u> of f at (x_0, y_0).

Geometrically, $f_x(x_0, y_0)$ is the slope of the tangent to the curve obtained by intersecting the surface $z = f(x, y)$ with the plane $y = y_0$ at $(x_0, y_0, f(x_0, y_0))$.

Physically, $f_x(x_0, y_0)$ is the rate of change in f at (x_0, y_0) along the x-axis

Computationally, $f_x(x_0, y_0)$ is determined usually by differentiating $f(x, y)$ w.r.t. x, treating y as a constant, and then substituting (x_0, y_0) for (x, y). More precisely, if $D_1 = \{ x \in \mathbb{R} : (x, y_0) \in D\}$ and $\varphi : D_1 \to \mathbb{R}$ is the function of one variable defined by

$$\varphi(x) = f(x, y_0).$$

Then φ is diff. at $x_0 \iff f_x(x_0, y_0)$ exists and in this case, $\varphi'(x_0) = f_x(x_0, y_0)$.

Likewise for $f_y(x_0, y_0)$.

As a consequence, partial derivatives of sums, scalar multiples, products, quotients possess exactly the same properties as the derivatives of functions of one variable.

One can also define left-hand and right-hand partial derivatives, denoted $(f_x)_-(x_0, y_0)$, $(f_x)_+(x_0, y_0)$, etc. in an obvious manner and these have all the "basic" properties.

Examples

① $f(x,y) = \sqrt{x^2+y^2}$ for $(x,y) \in \mathbb{R}$.

For $(x_0, y_0) \neq (0,0)$, we have

$$f_x(x_0,y_0) = \frac{x_0}{\sqrt{x_0^2+y_0^2}} \quad \text{and} \quad f_y(x_0,y_0) = \frac{y_0}{\sqrt{x_0^2+y_0^2}}$$

But $f_x(0,0)$ and $f_y(0,0)$ do not exist, since

$\lim_{t \to 0} \frac{|t|}{t}$ does not exist.

② $f(x,y) = \sin(xy)$ for $(x,y) \in \mathbb{R}$

Both the partial derivatives exist at every $(x_0,y_0) \in \mathbb{R}^2$;

$$f_x(x_0,y_0) = y_0 \cos x_0 y_0 \quad \& \quad f_y(x_0,y_0) = x_0 \cos(x_0 y_0)$$

Directional derivatives:

Let $\underline{u} = (u_1, u_2) \in \mathbb{R}^2$ be a unit vector, i.e.,

$$u_1^2 + u_2^2 = 1.$$

[Fixing $\underline{u}$ amounts to specifying a direction]

The directional derivative of a function $f : D \to \mathbb{R}$ at an interior point (x_0, y_0) of D in the direction of $\underline{u}$ is defined to be the limit

$$\lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t}$$

provided this limit exists. It is denoted by

$D_{\underline{u}} f(x_0, y_0)$.

Note that the partial derivatives are special cases of directional derivatives. Indeed

$$f_x(x_0,y_0) = D_{\underline{i}}f(x_0,y_0) \quad \& \quad f_y(x_0,y_0) = D_{\underline{j}}f(x_0,y_0)$$

where $\underline{i} = (1,0)$ and $\underline{j} = (0,1)$.

Examples: ① $f(x,y) = x^2+y^2$. For $t \neq 0$, look at

$[f(x_0+tu_1, y_0+tu_2) - f(x_0,y_0)]/t$

$= [(x_0+tu_1)^2 + (y_0+tu_2)^2 - x_0^2 - y_0^2]/t$

$= 2x_0u_1 + 2y_0u_2 + t \qquad (\because u_1^2+u_2^2=1)$

Hence $D_{\underline{u}}f(x_0,y_0)$ exists and is equal to $2x_0u_1+2y_0u_2$.

Observe that

$$D_{\underline{u}}f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \underline{u}.$$

(2) $f(x,y) = \sqrt{x^2+y^2}$ for $(x,y)\in\mathbb{R}^2$. For $t\neq 0$

$$\frac{\sqrt{(x_0+tu_1)^2+(y_0+tu_2)^2} - \sqrt{x_0^2+y_0^2}}{t}$$

$$= \frac{2x_0u_1+2y_0u_2+t}{\sqrt{(x_0+tu_1)^2+(y_0+tu_2)^2}+\sqrt{x_0^2+y_0^2}}$$

Thus if $(x_0,y_0)\neq(0,0)$, then $D_{\underline{u}}f(x_0,y_0)$ exists and

$$D_{\underline{u}}f(x_0,y_0) = \frac{x_0u_1+y_0u_2}{\sqrt{x_0^2+y_0^2}}$$

Again observe that

$$D_{\underline{u}}f(x_0,y_0) = \nabla f(x_0,y_0)\cdot\underline{u} \quad \text{for } (x_0,y_0)\neq(0,0).$$

3 $$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & \text{for } (x,y)\neq(0,0), \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$$

For a unit vector $\underline{u} = (u_1, u_2)$ and $t \neq 0$

$$\frac{f(0+tu_1, 0+tu_2) - f(0,0)}{t} = \frac{u_1^2 u_2}{u_1^4 t^2 + u_2^2}$$

Hence $D_{\underline{u}} f(0,0)$ exists and is given by

$$D_{\underline{u}} f(0,0) = \begin{cases} u_1^2/u_2 & \text{if } u_2 \neq 0, \\ 0 & \text{if } u_2 = 0. \end{cases}$$

In particular, $f_x(0,0) = 0 = f_y(0,0)$. [Recall, however, that f is not continuous at $(0,0)$] Observe here that $D_{\underline{u}} f(0,0) \neq \nabla f(0,0) \cdot \underline{u}$ unless $u_1 u_2 = 0$

(4) $f(x,y) = |x| + |y|$ for $(x,y) \in \mathbb{R}^2$

It is clear that f is continuous at $(0,0)$. On the other hand, if $\underline{u} = (u_1, u_2)$ is a unit vector & $t \neq 0$ then

$$\frac{f(0+tu_1, 0+tu_2) - f(0,0)}{t} = \frac{|t|}{t} \underbrace{(|u_1| + |u_2|)}_{\neq 0}$$

and so

$D_{\underline{u}} f(0,0)$ does not exist for ANY $\underline{u}$.

Exercises: Determine what happens when

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

and also the functions in the Tut Sheet No. 6 problems.

Differentiability

It may be noted that the definition of differentiability of a function of one variable does not extend in an obvious and straightforward manner to functions of several variables.

Observe, however, that for $D \subseteq \mathbb{R}$, $c \in D$ int. pt.,

$$f : D \to \mathbb{R} \text{ is diff. at } c \iff \exists\, \alpha \in \mathbb{R} \text{ s.t. } \lim_{h \to 0} \frac{f(c+h) - f(c) - \alpha h}{h} = 0.$$

With this in view, we make the following

Definition: Let $D \subseteq \mathbb{R}^2$ and (x_0, y_0) be an interior point of D. A function $f : D \to \mathbb{R}$ is said to be <u>differentiable</u> at (x_0, y_0) if $\exists\, (\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that

$$\lim_{(h,k) \to (0,0)} \frac{f(x_0+h, y_0+k) - f(x_0, y_0) - \alpha_1 h - \alpha_2 k}{\sqrt{h^2 + k^2}} = 0.$$

Note that if f is differentiable at (x_0, y_0) as per the above definition, then letting $(h,k) \to (0,0)$ along the x-axis or along the y-axis we obtain

$$\lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0) - \alpha_1 h}{h} = 0 = \lim_{k \to 0} \frac{f(x_0, y_0+k) - f(x_0,y_0) - \alpha_2 k}{k}$$

and thus if f is differentiable at (x_0, y_0), then BOTH $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist and

$$(\alpha_1, \alpha_2) = (f_x(x_0,y_0), f_y(x_0,y_0)) = \nabla f(x_0,y_0).$$

Food for thought (prelude to what will come next)

If f is differentiable at (x_0,y_0), then do all the directional derivatives of f at (x_0,y_0) exist? is f continuous at (x_0,y_0)? what could be an analogue of Carathéodory's Lemma?