MA 105 : Calculus

Sudhir R. Ghorpade

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai 400076, India
srg@math.iitb.ac.in
http://www.math.iitb.ac.in/∼srg/

IIT Goa

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Local, Absolute, and Constrained Extrema

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We have discussed the notions of

- local maxima/minima
- saddle point

for a real-valued function of two variables and noted the following

Necessary Condition [First Derivative Test]

Let $D \subseteq \mathbb{R}^2$ and (x_0, y_0) be an interior pt. of D.
Suppose $f : D \to \mathbb{R}$ is such that ∇f exists at (x_0, y_0).
Then f has a local extrema or a saddle point at (x_0, y_0)
$\Longrightarrow$ $\nabla f(x_0, y_0) = (0,0)$, i.e., $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$.

Next we defined (for a function $f : D \to \mathbb{R}$ with continuous first & second order partials at (x_0, y_0)) an important quantity, namely the discriminant of f at (x_0, y_0), viz.,

$$\Delta f(x_0,y_0) = f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2$$

$$= \begin{vmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{yx}(x_0,y_0) & f_{yy}(x_0,y_0) \end{vmatrix}$$

With this at our disposal, we have the following useful

Sufficient Condition [Discriminant Test]

Let $f : D \to \mathbb{R}$ and $(x_0, y_0) \in D$ be as above.
Suppose $\nabla f(x_0, y_0) = (0,0)$. Then:

(i) $\Delta f(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) < 0 \implies f$ has a local maximum at (x_0,y_0)

(ii) $\Delta f(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) > 0 \implies f$ has a local minimum at (x_0,y_0).

(iii) $\Delta f(x_0,y_0) < 0 \implies f$ has a saddle point at (x_0,y_0).

Remark: If $\Delta f(x_0, y_0) = 0$, then the discriminant test is inconclusive. In this case, either of the 3 possibilities can occur, provided of course $\nabla f(x_0, y_0) = (0, 0)$. For example,

- $f(x,y) = -(x^4 + y^4)$ has loc. max. at $(0,0)$
- $f(x,y) = x^4 + y^4$ has loc. min. at $(0,0)$
- $f(x,y) = x^3 y^3$ has a saddle point at $(0,0)$:

$$3x^2y^3 \qquad\qquad 3x^3y^2 \qquad \text{first order partials}$$

$$6xy^3 \qquad 9x^2y^2 \qquad 6x^3y \qquad \text{second order partials}$$

Absolute Extremum

Recall that

if D is a closed & bounded subset of $\mathbb{R}^2$ and $f : D \longrightarrow \mathbb{R}$ is continuous, then f is bounded and attains its maximum & minimum.

In this case, we have

$$M = \sup\{f(x,y) : (x,y) \in D\} = f(c_1, d_1) \quad \text{and}$$

$$m = \inf\{f(x,y) : (x,y) \in D\} = f(c_2, d_2)$$

for some $(c_1, d_1), (c_2, d_2) \in D$. To determine these absolute extrema and the points where they are attained, we may proceed as follows.

Definition : An interior point (x_0, y_0) of D is called a critical point of $f : D \to \mathbb{R}$ if

- either $\nabla f(x_0, y_0)$ exists and is $(0,0)$
- or $\nabla f(x_0, y_0)$ does not exist.

From the Necessary condition for local extrema, we immediately obtain :

Theorem : The absolute maximum as well as the absolute minimum of f is attained either at a critical point of f or at a boundary point of D.

With this in view, we deduce the following

Recipe for finding absolute extrema :

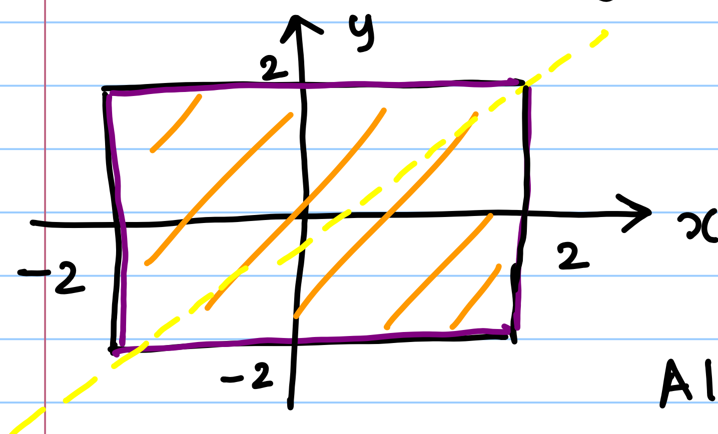
- Find the boundary of D and the absolute extrema of f on the boundary of D
  (this is usually a "one-variable problem")
- Find the critical points of f and the values of f at these points
- Compare these values
  - the biggest gives the absolute maximum
    and the smallest " " absolute minimum.

Examples: ① $D = \overline{S}_2(0,0) = \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{l} |x| \le 2, \\ |y| \le 2 \end{array} \right\}.$

Consider $f : D \to \mathbb{R}$ given by $f(x,y) = 4xy - 2x^2 - y^4$.

The boundary $\longleftrightarrow$ $x = 2$ or -2

or $y = 2$ or -2



We have

$$f(2,y) = 8y - 8 - y^4, \quad y \in [-2,2]$$

and this has absolute maximum at $y = \sqrt[3]{2}$ and abs min at $y = -2$.

Also

$$f(x,-2) = -8x - 2x^2 - 16, \quad x \in [-2,2]$$

and this has abs. max at $x = -2$ and abs min at $x = 2$.

Since $f(x,y) = f(-x,-y)$, we need not consider $f(-2,y)$ and $f(x,2)$.

Next find the critical points of f:

$$f(x,y) = 4xy - 2x^2 - y^4 \implies \begin{aligned} f_x &= 4y - 4x \\ f_y &= 4x - 4y^3 \end{aligned}$$

Thus

$$\nabla f(x,y) = (0,0) \implies (x,y) = (0,0),\ (1,1),\ (-1,-1).$$

and so $(0,0)$, $(\pm 1, \pm 1)$ are the only critical points.

Finally, compare the values:

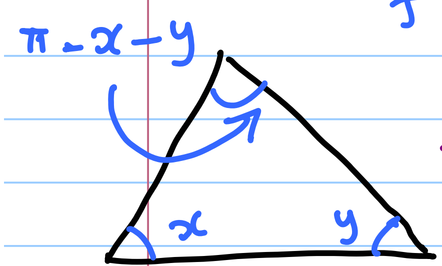
(x,y)	$(0,0)$	$(\pm 1, \pm 1)$	$(\pm 2, \pm \sqrt[3]{2})$	$(\pm 2, \mp 2)$	$(\pm 2, \pm 2)$
$f(x,y)$	0	1	$6\sqrt[3]{2} - 8$	-40	-8

Hence we conclude that the absolute maximum of f is 1, attained at $(1,1)$ & $(-1,-1)$ while the absolute minimum of f is -40 attained at $(2,-2)$ & $(-2,2)$.

② Find the triangle for which the product of sines of the three angles is the largest

$\Longleftrightarrow$ finding the absolute maximum of

$f(x,y) = (\sin x)(\sin y)\sin(x+y)$, $0 \le x, y, x+y \le \pi$.



Boundary: If x, y, or $x+y$ is 0 or π, then $f(x,y) = 0$.

Critical points: For $0 < x, y, x+y < \pi$,

$$\nabla f = (0,0) \implies \cos x \underbrace{\sin y}_{\neq 0} \sin(x+y) + \sin x \underbrace{\sin y}_{\neq 0} \cos(x+y) = 0$$

$$\underbrace{\sin x}_{\neq 0} \cos y \sin(x+y) + \underbrace{\sin x}_{\neq 0} \sin y \cos(x+y) = 0$$

$$\implies \sin(2x+y) = 0 = \sin(2y+x)$$

$$\implies 2x+y = \pi = 2y+x \qquad \left(\because 0 < \begin{matrix} 2x+y, \\ 2y+x \end{matrix} < 2\pi\right)$$

$$\implies x = y = \pi/3. \quad \text{Also } f(\pi/3, \pi/3) > 0.$$

Thus f has abs max at $(\frac{\pi}{3}, \frac{\pi}{3})$ and the desired triangle is equilateral.

Constrained Maxima and Minima

Let $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a function. Finding the absolute extrema of f on the boundary of D often reduces to the following

Problem: Find the absolute maximum/minimum of
$$f(x,y)$$
subject to the constraint
$$g(x,y) = 0.$$

If one can solve the constraint equation $g(x,y)=0$ in one of the variables, then this reduces to a one-variable problem. But there is also another way that is nice and elegant:

Lagrange's Method (of Undetermined Multipliers):

This method is based on the following

Lagrange Multiplier Theorem (LMT):

Let $D \subseteq \mathbb{R}^2$, (x_0, y_0) interior point of D, $f, g : D \to \mathbb{R}$ functions having continuous partial derivatives in $S_r(x_0, y_0)$ for some $r > 0$. Suppose

- $g(x_0, y_0) = 0$
- $\nabla g(x_0, y_0) \neq (0,0)$
- f, when restricted to $C = \{(x,y) \in D : g(x,y) = 0\}$, has an absolute extremum at (x_0, y_0)

THEN

$$\nabla f(x_0, y_0) = \lambda_0 \nabla g(x_0, y_0) \text{ for some } \lambda_0 \in \mathbb{R}.$$

Remark: The third condition in LMT is usually deduced easily when C is closed and bounded, and f is continuous.

Proof of LMT (Sketch): Suppose $g_y(x_0, y_0) \neq 0$.
By the implicit function theorem, we can find $\delta > 0$ and a function $y : [x_0 - \delta, x_0 + \delta] \to \mathbb{R}$ such that

$$y(x_0) = y_0, \quad g(x, y(x)) = 0 \quad \forall\, x \in [x_0 - \delta, x_0 + \delta]$$

$$\Rightarrow \quad g_x(x_0, y_0) + y'(x_0)\, g_y(x_0, y_0) = 0. \quad \text{— (A)}$$

Now the last condition implies that $\varphi : [x_0 - \delta, x_0 + \delta] \to \mathbb{R}$ defined by $\varphi(x) = f(x, y(x))$ has abs. extremum at x_0.
Hence

$$0 = \varphi'(x_0) = f_x(x_0, y_0) + y'(x_0)\, f_y(x_0, y_0) = 0 \quad \text{— (B)}$$

Comparing (A) and (B) we find $\nabla f(x_0, y_0) = \lambda_0 \nabla g(x_0, y_0)$ where $\lambda_0 = (f_y / g_y)(x_0, y_0)$. The case $g_x(x_0, y_0) \neq 0$ is similar.

The LMT gives us the following nice

Lagrange's METHOD of Undetermined Multipliers:

To find the absolute max/min of $f(x, y)$ subject to the constraint $g(x, y) = 0$,

- introduce a new variable λ (the undetermined multiplier)
- find simultaneous solutions of

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = 0$$

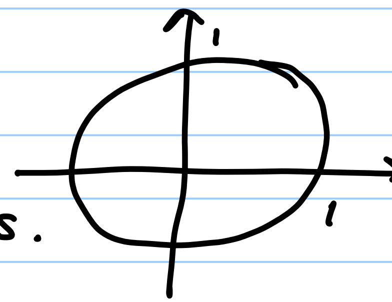
at which $\nabla g(x, y) \neq (0, 0)$.

- compare the values of f at these simultaneous solutions to conclude (by LMT) the absolute maximum/minimum of f.

[points where the hypothesis of LMT doesn't hold, e.g., where $\nabla g = (0,0)$, have to be considered separately.]

Examples : ① Find the maximum of $f(x,y) = xy$ on the unit circle.

Here
$$g(x,y) = x^2 + y^2 - 1, \qquad (x,y) \in \mathbb{R}^2$$
and the constraint set
$$C = \{ (x,y) \in \mathbb{R}^2 : g(x,y) = 0 \}$$



is closed & bounded and f is continuous. So f does have an abs. max. on C.

Now look at $\quad \nabla f = \lambda \nabla g \quad$ and $\quad g = 0$

i.e., $\quad (y, x) = \lambda (2x, 2y) \quad$ and $\quad x^2 + y^2 - 1 = 0$

$\implies \quad x^2 = y^2 \qquad$ and $\quad x^2 + y^2 - 1 = 0$

$\implies \quad (x,y) = \left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}} \right), \left(\pm \frac{1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}} \right)$

Comparing values, we see that the absolute maximum of f is $1/2$, attained at $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$

The Case of Three Variables

There is a straightforward analogue of the LMT for functions of three variables where $\mathbb{R}^2$ is replaced by $\mathbb{R}^3$, (x_0, y_0) by (x_0, y_0, z_0) and the constraint set is now a surface, viz.

$$\{(x, y, z) \in D : g(x, y, z) = 0\}$$

rather than a curve C.

This leads to the corresponding Lagrange's method of undetermined multipliers where to find abs. max/min of $f(x, y, z)$ subject to $g(x, y, z) = 0$, we seek simultaneous solutions of

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \quad \text{and} \quad g(x, y, z) = 0$$

at which $\nabla g(x, y, z) \neq (0, 0, 0)$.

Examples: 1 Find the maximum of

$$f(x,y,z) = x^2y^2z^2$$

on the unit sphere.

Here the constraint is

$$g(x,y,z) := x^2+y^2+z^2-1 = 0.$$

Consider

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \quad 2xy^2z^2 = 2\lambda x, \quad 2x^2yz^2 = 2\lambda y, \quad 2x^2y^2z = 2\lambda z$$

If one of x, y, z is 0, then $f = 0$ at that pt. Otherwise:

$$x^2 = y^2 = z^2 \text{ and } \lambda = x^4.$$

If in addition $g(x,y,z) = 0$, then $x = y = z = \pm \frac{1}{\sqrt{3}}$.

Also $\{g = 0\}$ is closed & bounded and f is cont.

So the (absolute) maximum of f is $\frac{1}{27}$ attained at, for instance $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

(2) Find the points on the surface $z^2 = xy + 4$ closest to the origin.

$\longleftrightarrow$ finding absolute minimum of

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to

$$g(x, y, z) = xy + 4 - z^2 = 0 .$$

Consider

$$\nabla f = \lambda \nabla g$$

$$\Longrightarrow \quad 2x = \lambda y, \quad 2y = \lambda x, \quad 2z = -2\lambda z$$

If $\lambda \neq 0$ (for otherwise $x = y = z = 0$ but $g(0,0,0) \neq 0$), then $x = 0$ or $\lambda = \pm 2$. One sees that the simultaneous solutions of $\nabla f = \lambda \nabla g$ and $g = 0$ are $(0,0,2)$, $(0,0,-2)$, $(2,-2,0)$, $(-2,2,0)$. Comparing values, one sees that the minimum of f is 4 and the points $(0, 0, \pm 2)$ on the surface are closest to the origin.

The Case of two (or more) constraints

If instead of a single constraint such as

$$g(x,y,z)=0$$

one has two constraints, say

$$g(x,y,z)=0, \text{ and}$$

$$h(x,y,z)=0$$

then a variant of LMT shows that to find the absolute max/min, one seeks simultaneous solutions of

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ \text{and the constraints } g=0,\ h=0 \end{cases}$$

in the unknowns x, y, z, λ, μ at which we have

$\nabla g \neq 0$, $\nabla h \neq 0$ and ∇g not parallel to ∇h.

Example : Find points on the intersection of the planes $x+y+z=1$ and $3x+2y+z=6$ that are closest to the origin.

$\longleftrightarrow$ finding abs. min. of

$$f(x,y,z) = x^2+y^2+z^2$$

subject to the two constraints

$$\begin{cases} g(x,y,z) = x+y+z-1 = 0 \\ h(x,y,z) = 3x+2y+z-6 = 0 \end{cases}$$

We look at

$$\nabla f = \lambda \nabla g + \mu \nabla h \quad \text{and} \quad g=h=0$$

$\Longrightarrow$ $x = \dfrac{\lambda+3\mu}{2}$, $y = \dfrac{\lambda+2\mu}{2}$, $z = \dfrac{\lambda+\mu}{2}$ & $g=h=0$

$\Longrightarrow$ $3\lambda+6\mu=2$ and $3\lambda+7\mu=6$.

This gives

$$\lambda = -22/3 \quad \text{and} \quad \mu = 4$$

and hence

$$(x, y, z) = \left(\frac{7}{3}, \frac{1}{3}, \frac{-5}{3} \right)$$

Also $\nabla g = (1,1,1)$ and $\nabla h = (3,2,1)$ are both nonzero and are not parallel. Hence the desired point is $\left(\frac{7}{3}, \frac{1}{3}, \frac{-5}{3} \right)$.

[Question: why does f have minimum here and not the maximum here?]

Appendix : Proof of the Discriminant Test

Since f_{xx}, f_{yy} and f_{xy} are continuous in $S_r(x_0, y_0)$, the discriminant

$$\Delta f = f_{xx} f_{yy} - f_{xy}^2$$

is continuous in $S_r(x_0, y_0)$ and hence when $\Delta f(x_0, y_0) \neq 0$, there is $\delta > 0$ such that for every $(x,y) \in S_\delta(x_0, y_0)$, $\Delta f(x,y) \neq 0$ and it has the same sign as $\Delta f(x_0, y_0)$. Moreover, we may also choose $\delta > 0$ such that $f_{xx}(x,y)$ also has the same sign as $f_{xx}(x_0, y_0)$ for every $(x,y) \in S_\delta(x_0, y_0)$.

Now by Taylor's theorem, for any $(x,y) \in S_\delta(x_0,y_0)$

$$f(x,y) - f(x_0,y_0) = 0 \cdot h + 0 \cdot k + \frac{1}{2}\left[A h^2 + 2Bhk + C k^2\right]$$

where

$$A = f_{xx}(c,d), \quad B = f_{xy}(c,d), \quad C = f_{yy}(c,d)$$

for some $(c,d) \in S_\delta(x_0,y_0)$, $h = x - x_0$, $k = y - y_0$.

Observe that

$$A\left[A h^2 + 2Bhk + C k^2\right] = (Ah + Bk)^2 + \Delta k^2$$

where

$$\Delta = AC - B^2 = \Delta f(c,d).$$

It follows that

$$\Delta f(x_0,y_0) > 0 \ \& \ f_{xx}(x_0,y_0) > 0 \implies f(x,y) - f(x_0,y_0) \geq 0,$$

&

$$\Delta f(x_0,y_0) > 0 \ \& \ f_{xx}(x_0,y_0) < 0 \implies f(x,y) - f(x_0,y_0) \leq 0.$$

Since (x, y) is an arbitrary element of $S_\delta(x_0, y_0)$, parts (i) and (ii) of the discriminant test are proved. Finally if $\Delta f(x_0, y_0) < 0$ [and so $\Delta f < 0$ at every point of $S_\delta(x_0, y_0)$], then

$$f(x, y) - f(x_0, y_0) = \frac{k^2}{2} Q(h/k) \quad \text{for} \quad \begin{array}{l} |h| < \delta \\ 0 < |k| < \delta \end{array}$$

where $Q(t)$ is the quadratic $At^2 + 2Bt + C$. Since $AC - B^2 < 0$, we have $(2B)^2 - 4AC > 0$

and so if $A \neq 0$, then $Q(t)$ has two distinct real roots and there are some $t_1, t_2 \in \mathbb{R}$ such that $Q(t_1) < 0$ and $Q(t_2) > 0$. So we can take (h, k) to be $(\delta t_1/2, \delta/2)$ or $(\delta t_2/2, \delta/2)$ to find $(x_1, y_1), (x_2, y_2)$

in $S_\delta(x_0, y_0)$ such that

$$f(x_1, y_1) < f(x_0, y_0) \quad \text{and} \quad f(x_2, y_2) > f(x_0, y_0). \qquad (*)$$

Hence f has a saddle point at (x_0, y_0).

In case $AC - B^2 < 0$ and $A = 0$, we have

$$f(x, y) - f(x_0, y_0) = \frac{k}{2}(2Bh + Ck)$$

Now taking (h, k) equal to $(0, \delta')$ or $(C\delta', -B\delta')$ for a suitably small positive value of δ', we see that the RHS above is $\frac{C\delta'^2}{2}$ or $-B^2C\delta'^2$, which have opposite signs if $C \neq 0$. So again $(*)$ holds for suitable (x_1, y_1), (x_2, y_2). Finally if $A = C = 0$, then it suffices to take (h, k) to be $(\delta/2, \delta/2)$ or $(\delta/2, -\delta/2)$. $\square$