MA 105 : Calculus

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Autumn 2016

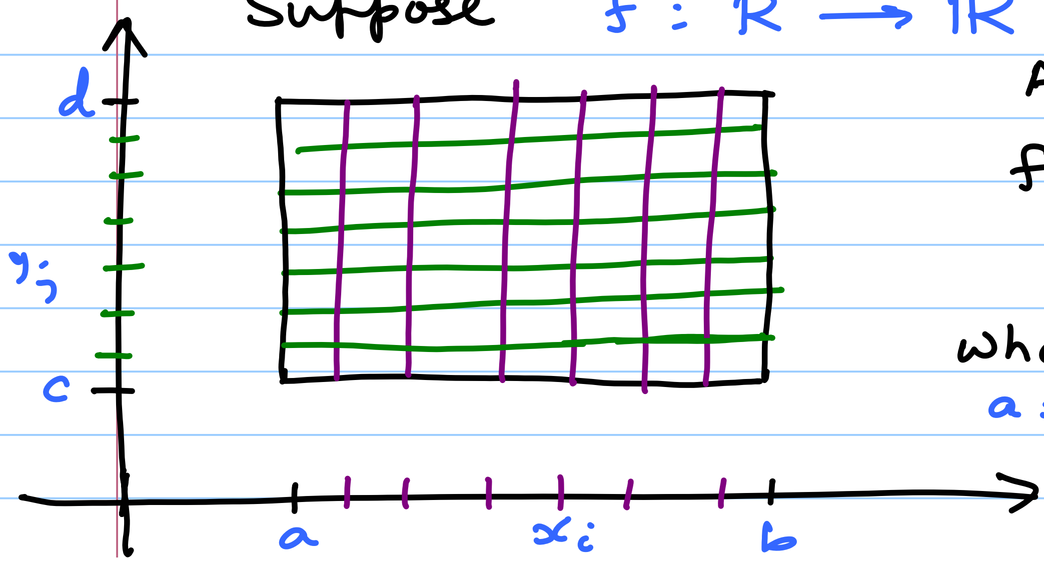
Note Title 10/4/2011

Double Integrals

First, we extend Riemann integral to bounded functions defined on rectangles

$R = [a, b] \times [c, d] = \{ (x, y) \in \mathbb{R}^2 : a \le x \le b \ \& \ c \le y \le d \}$.

Suppose $f : R \longrightarrow \mathbb{R}$ is bounded



A partition of R is a finite set

$P = \{ (x_i, y_j) : \begin{array}{l} i = 0, 1, \ldots, n \\ j = 0, 1, \ldots, k \end{array} \}$

where

$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$

& $c = y_0 < y_1 < \cdots < y_{k-1} < y_k = d.$

Define the <u>mesh</u> of P to be

$$\mu(P) = \max\{x_1 - x_0, \ldots, x_n - x_{n-1}, y_1 - y_0, \ldots, y_k - y_{k-1}\}.$$

and the lower and upper Riemann sums of f w.r.t. P to be

$$L(P, f) = \sum_{i=1}^{n} \sum_{j=1}^{k} m_{ij}(f) A_{ij}$$

$$U(P, f) = \sum_{i=1}^{n} \sum_{j=1}^{k} M_{ij}(f) A_{ij}$$

where

$$A_{ij} = \text{area of } (i,j)^{th} \text{ subrectangle } R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$
$$= (x_i - x_{i-1})(y_j - y_{j-1})$$

and

$$m_{ij}(f) = \inf\{ f(x,y) : (x,y) \in R_{ij} \}$$

$$M_{ij}(f) = \sup\{ f(x,y) : (x,y) \in R_{ij} \}$$

Observe that if $m \leq f(x,y) \leq M$, $\forall (x,y) \in R$, then

$$m(b-a)(d-c) \leq L(P,f) \leq U(P,f) \leq M(b-a)(d-c)$$

We define

$$L(f) = \sup\{ L(P,f) : P \text{ partition of } R \}$$

$$U(f) = \inf\{ U(P,f) : \quad " \quad \}$$

and we say f is (double) integrable on R if

$$L(f) = \bar{U}(f).$$

The common value is called the double integral of f on R and denoted by

$$\iint_R f(x,y)\,d(x,y) \quad \text{or simply} \quad \iint_R f.$$

Definition: If $f: R \to \mathbb{R}$ is integrable & nonnegative, then the volume of the solid under the surface $z = f(x,y)$, $(x,y) \in R$, is defined to be $\iint_R f$.

Example : ① $f(x,y) = r \quad \forall (x,y) \in R = [a,b] \times [c,d]$

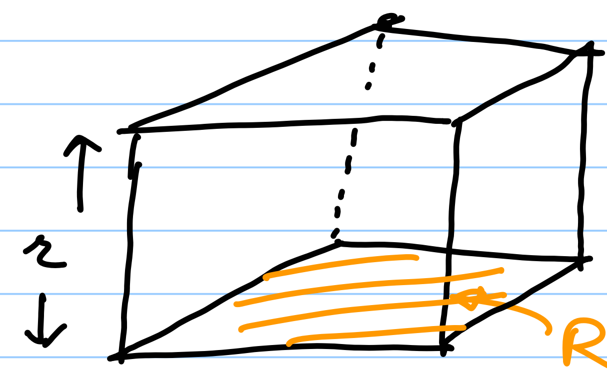
— a constant function.

Note that if $P = \{ (x_i, y_j) : i = 0, 1, \dots, n, \; k = 0, 1, \dots, k \}$ is any partition of R, then

$$L(P, f) = \sum_{i=1}^{n} \sum_{j=1}^{k} r A_{ij} = r(b-a)(d-c) = U(P, f).$$

Hence f is integrable and

$$\iint_R f = r\,(b-a)\,(d-c).$$



Note that if $r \geq 0$, then the double integral is the volume of the box with base R and height r.

(2) Suppose $f : R \longrightarrow \mathbb{R}$ is the bivariate Dirichlet function defined by

$$f(x,y) = \begin{cases} 1 & \text{if both } x, y \text{ are rational,} \\ 0 & \text{if } x \text{ or } y \text{ is irrational.} \end{cases}$$

Then for any partition P of R,

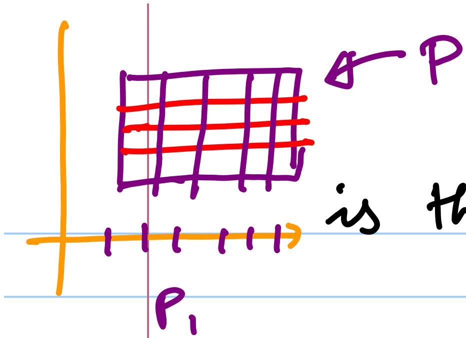
$$L(P,f) = 0 \quad \text{and} \quad U(P,f) = (b-a)(d-c)$$

Hence f is not integrable (since $L(f) = 0 \neq 1 = U(f)$!)

(3) $f(x,y) = \varphi(x)$ for $(x,y) \in R = [a,b] \times [c,d]$ where $\varphi : [a,b] \longrightarrow \mathbb{R}$ is Riemann-integrable.

Note that if $P = \{ (x_i, y_j) : i = 0, 1, \ldots, n,\ j = 0, 1, \ldots, k \}$ is any partition of R and if

$$P_1 = \{ a = x_0, x_1, \ldots, x_n = b \}$$



is the corresponding partition of $[a,b]$, then

$$m_{i,j}(f) = m_i(\varphi) \;\; \& \;\; M_{ij}(f) = M_i(\varphi)$$

and hence $L(P,f) = (d-c)\,L(P_1,\varphi)$ and $U(P,f) = (d-c)\,U(P_1,\varphi)$. It follows that f is integrable on R and $\iint_R f = (d-c)\int_a^b \varphi(x)\,dx$.

A similar conclusion holds in case $f(x,y) = \psi(y)$ where $\psi : [c,d] \to \mathbb{R}$ is Riemann-integrable.

Riemann Condition: Let $f : R \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\varepsilon > 0$, $\exists$ a partition P_ε of R such that

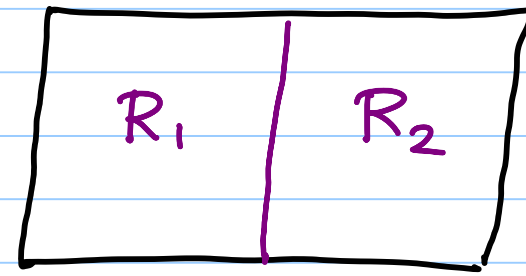
$$U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon.$$

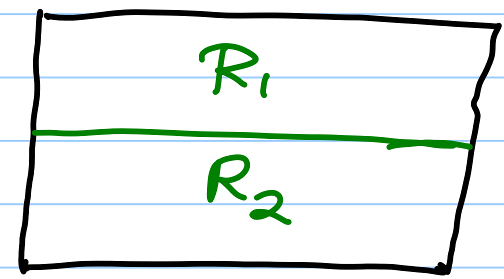
Domain additivity: If the closed rectangle

$$R = [a,b] \times [c,d]$$

is a nonoverlapping union of two closed rectangles

$$R = R_1 \cup R_2$$





then

f is integrable on R

$\iff$ f is integrable on R_1 as well as on R_2

and in this case

$$\iint_R f = \iint_{R_1} f + \iint_{R_2} f .$$

Basic fact: If $f : R \to \mathbb{R}$ is continuous, then f is integrable

Algebraic and Order Properties:

1. If $f, g : R \to \mathbb{R}$ are integrable, then

   $$f+g, \quad rf, \quad \text{and} \quad fg \qquad (r \in \mathbb{R})$$

   are integrable and

   $$\iint_R f+g = \iint_R f + \iint_R g \quad \text{and} \quad \iint_R rf = r \iint_R f.$$

2. If $f : R \to \mathbb{R}$ is integrable and "bounded away from zero", i.e., $\exists\, \delta > 0$ s.t. $|f(x,y)| \geq \delta$ for all $(x,y) \in R$, then $\frac{1}{f} : R \to \mathbb{R}$ is integrable.

3. $f, g : R \to \mathbb{R}$ integrable and $f \leq g$ on $R \Longrightarrow \iint_R f \leq \iint_R g$.

Fubini's Theorem on Rectangles

Let $f : R \to \mathbb{R}$ be integrable.

(i) If for each fixed $x \in [a,b]$, the Riemann integral $A(x) = \int_c^d f(x,y)\,dy$ exists,

then the iterated integral

$$\int_a^b A(x)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

exists and is equal to $\iint_R f(x,y)\,d(x,y)$.

(ii) If for each fixed $y \in [c,d]$, the Riemann integral $B(y) = \int_a^b f(x,y)\,dx$ exists, then the other iterated integral

$$\int_c^d B(y)\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy \quad \text{exists}$$

and is equal to $\iint_R f(x,y)\,d(x,y)$.

(iii) If the hypotheses in (i) & (ii) hold [in particular, if f is continuous on R], then

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \iint_R f = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

Caution : Remember that f needs to be integrable for Fubini's Theorem to be applicable.

Examples : 1 $f(x,y) = \varphi(x)\psi(y)$ for $(x,y) \in R$

where $\varphi : [a,b] \to \mathbb{R}$ and $\psi : [c,d] \to \mathbb{R}$ are Riemann-integrable functions of one variable.

Then f is integrable on R (by Ex. 3 and since products of integrable functions are integrable). Thus by Fubini's Theorem

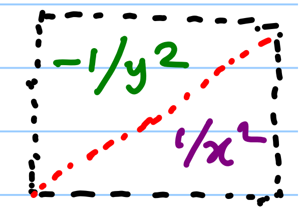
$$\iint_R f = \int_a^b \left(\int_c^d \varphi(x)\psi(y)\,dy \right) dx = \left(\int_a^b \varphi(x)\,dx \right) \left(\int_c^d \psi(y)\,dy \right).$$

In particular,

$$\iint_R x^r y^s \, d(x,y) = \left(\frac{b^{r+1} - a^{r+1}}{r+1} \right) \left(\frac{d^{s+1} - c^{s+1}}{s+1} \right) \quad \forall\, r,s \geq 0.$$

(2) Consider $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 1/x^2 & \text{if } 0 < y < x < 1, \\ -1/y^2 & \text{if } 0 < x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$



Then f is not bounded on $[0,1] \times [0,1]$ and hence not integrable. On the other hand, for $0 < x < 1$

$$A(x) = \int_0^1 f(x,y)\,dy = \int_0^x \frac{1}{x^2}\,dy + \int_x^1 -\frac{1}{y^2}\,dy = \frac{(x-0)}{x^2} + \left[\frac{1}{y}\right]_x^1 = 1.$$

whereas $A(0) = 0 = A(1)$. Hence $A : [0,1] \to \mathbb{R}$ is Riemann integrable and

$$\int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx = \int_0^1 A(x)\,dx = \int_0^1 1\,dx = 1.$$

Similarly, if $0 < y < 1$, then

$$B(y) = \int_0^1 f(x,y)\,dx = \int_0^y \frac{-1}{y^2}\,dx + \int_y^1 \frac{1}{x^2}\,dx$$

$$= \frac{-1}{y^2}(y-0) + \left[\frac{-1}{x}\right]_y^1 = -1,$$

whereas $B(0) = 0 = B(1)$. Thus B is Riemann-integrable on $[0,1]$ and

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy = \int_0^1 B(y)\,dy = \int_0^1 -1\,dy = -1.$$

This example shows that the two iterated integrals may exist, but they need not be equal when f is not integrable.

Remark: In [GL-2, p. 221], you can find an example of an integrable function

$$f : [0,1] \times [0,1] \to \mathbb{R}$$

for which only one of the two iterated integrals exists. This shows the importance of the hypothesis that $\int_c^d f(x,y)\,dy$ exists or that $\int_a^b f(x,y)\,dx$ exists in parts (i) and (ii) of Fubini's theorem on rectangles.

(Approximate) Riemann double sums

For a partition $P = \{(x_i, y_j) : i = 0, 1, \dots, n,\ j = 0, 1, \dots, k\}$ of $R = [a,b] \times [c,d]$ and a bounded function $f : R \to \mathbb{R}$ any sum of the form

$$S(P, f) = \sum_{i=1}^{n} \sum_{j=1}^{k} f(\xi_{ij})\, A_{ij}$$

where ξ_{ij} is an arbitrary element of the $(i,j)^{th}$ subrectangle $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ of P and $A_{ij} = \text{area}(R_{ij}) = (x_i - x_{i-1})(y_j - y_{j-1})$, is called a Riemann double sum for f w.r.t. P.

Theorem of Darboux :

Let $f : R \to \mathbb{R}$ be a bounded function and let $I \in \mathbb{R}$. Then

f is integrable on R and $I = \iint_R f$ $\iff$ $\lim\limits_{\mu(P)\to 0} S(P,f) = I$,

i.e., for any $\varepsilon > 0$, $\exists\, \delta > 0$ such that

P partition of R & $\mu(P) < \delta \implies |S(P,f) - I| < \varepsilon$

where $S(P,f)$ is any Riemann double sum for f w.r.t. P.

Remark: For proofs of this and the other results stated earlier, one may refer to [GL-2]. This is an optional activity!

Double integrals over bounded sets

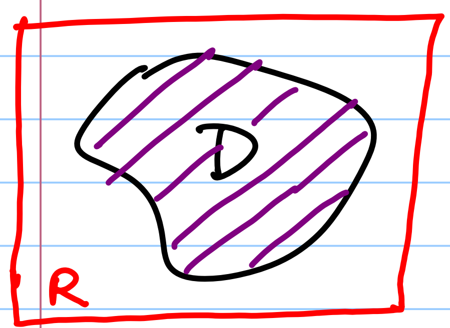
Let D be a bounded subset of $\mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a bounded function.

To define a suitable notion of integrability and that of $\iint_D f$, we may proceed as follows.

Consider a rectangle R containing D and the function

$$f^* : R \to \mathbb{R}$$

defined by



$$f^*(x, y) = \begin{cases} f(x, y) & \text{if } (x,y) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly f^* is a bounded function on R.

Definition : $f : D \to \mathbb{R}$ is <u>integrable</u> over D if $f^* : R \to \mathbb{R}$ is integrable on R. In this case, we define

$$\iint_D f(x,y)\, d(x,y) = \iint_R f^*(x,y)\, d(x,y).$$

This is called the <u>double integral</u> of f (over D).

Remark : This definition is independent of the choice of a rectangle R containing D and is consistent with our previous defn. of double integrals on rectangles. This follows from the domain additivity of double integrals on rectangles.

In case f is integrable on D and nonnegative, we define

$$\mathrm{Vol}(E_f) = \text{volume of the solid under the surface } z = f(x,y),\ (x,y) \in D$$

$$= \iint_D f = \iint_D f(x,y)\, d(x,y),$$

where

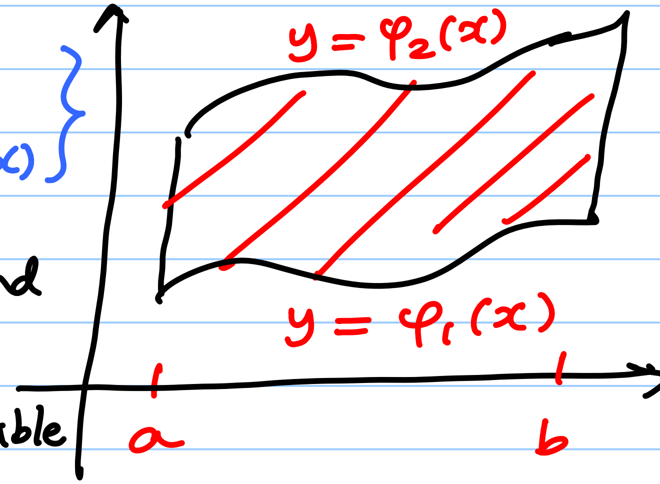
$$E_f = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D \text{ and } 0 \le z \le f(x,y)\}.$$

The algebraic and order properties readily extend from double integrals on rectangles to double integrals over bounded domains.

Fubini's Theorem over Elementary Regions

Let D be a bounded subset of $\mathbb{R}^2$. If

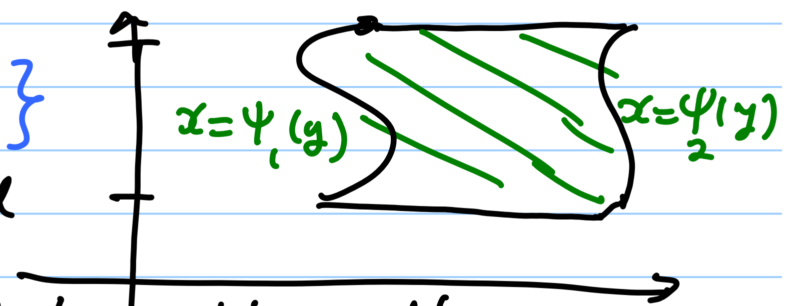
$$D = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b \ \& \ \varphi_1(x) \leq y \leq \varphi_2(x)\}$$



for some $a, b \in \mathbb{R}$ with $a < b$, and $\varphi_1, \varphi_2 : [a,b] \to \mathbb{R}$ such that $\varphi_1 \leq \varphi_2$ and φ_1, φ_2 are Riemann integrable

OR if

$$D = \{(x,y) \in \mathbb{R}^2 : c \leq y \leq d \text{ and } \psi_1(y) \leq x \leq \psi_2(y)\}$$



for some $c, d \in \mathbb{R}$ with $c < d$, and $\psi_1, \psi_2 : [c,d] \to \mathbb{R}$ such that $\psi_1 \leq \psi_2$ and ψ_1, ψ_2 are Riemann-integrable, then D is said to be an <u>elementary region.</u>

Fubini's Theorem over Elementary Regions:

If D is an elementary region as above and if $f : D \to \mathbb{R}$ is integrable, then

$$\iint_D f = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy \right) dx$$

provided the inner Riemann integral $\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy$ exists for each fixed $x \in [a,b]$.

Likewise,

$$\iint_D f = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx \right) dy$$

provided the inner Riemann integral $\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx$ exists for each fixed $y \in [c,d]$.

Examples: 1 [Tut 8, Q 1(i)]

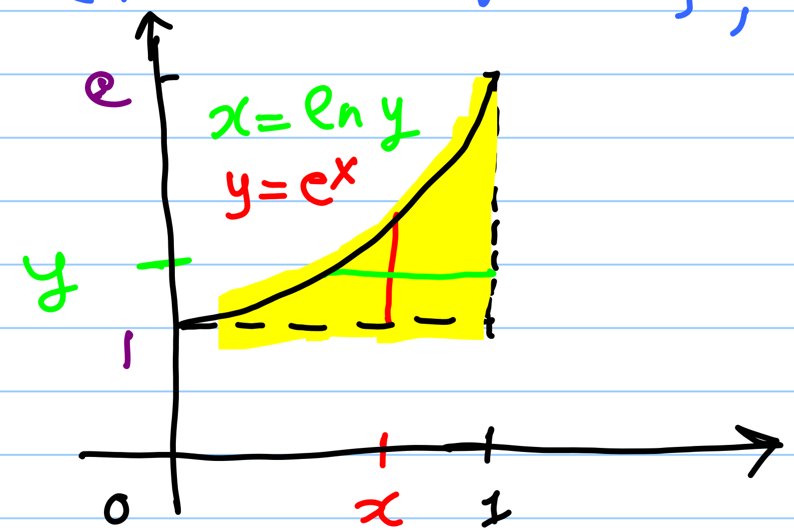
$$\int_0^1 \left(\int_1^{e^x} dy \right) dx = \iint_D d(x,y) = \int_1^e \left(\int_{\ln y}^1 dx \right) dy$$

where

$$D = \{ (x,y) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ and } 1 \le y \le e^x \},$$

using Fubini's Theorem.

Note that D is an elementary region of type I as well as type II.

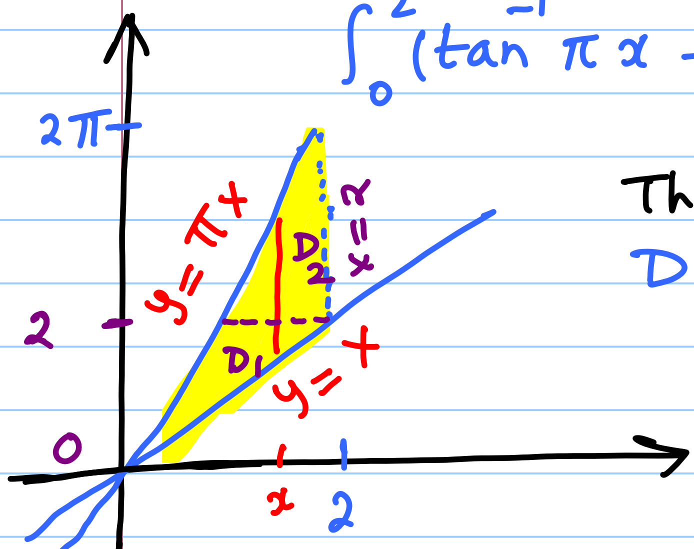


② Evaluate $\displaystyle\int_0^2 (\tan^{-1}\pi x - \tan^{-1} x)\,dx$.

[Tut 8, Q. 2(iii)]

This is a Riemann integral, which is not easy to evaluate directly. However, converting it to a double integral and using Fubini's Theorem helps. Indeed,

$$\int_0^2 (\tan^{-1}\pi x - \tan^{-1} x)\,dx = \int_0^2 \left(\int_x^{\pi x} \frac{1}{1+y^2}\,dy \right) dx$$



The RHS is the double integral of $\frac{1}{1+y^2}$ over

$D = \{ (x,y) : 0 \le x \le 2 \text{ and } x \le y \le \pi x \}$

$= D_1 \cup D_2$ where $D_1 = \{(x,y) : 0 \le y \le 2,\ \frac{y}{\pi} \le x \le y\}$

and $D_2 = \{ (x,y) : 2 \le y \le 2\pi,\ \frac{y}{\pi} \le x \le 2\}$

and so by Fubini's theorem,

the given integral equals

$$\int_0^2 \left(\int_{y/\pi}^{y} \frac{1}{1+y^2}\, dx \right) dy + \int_2^{2\pi} \left(\int_{y/\pi}^{2} \frac{1}{1+y^2}\, dx \right) dy$$

$$= \left(1 - \frac{1}{\pi}\right) \int_0^2 \frac{y\, dy}{1+y^2} + \int_2^{2\pi} \frac{2\, dy}{1+y^2} - \frac{1}{\pi} \int_2^{2\pi} \frac{y\, dy}{1+y^2}$$

$$= \frac{1}{2}\left(1 - \frac{1}{\pi}\right) \left[\ln (1+y^2) \right]_0^2 + 2 \left[\tan^{-1} y \right]_2^{2\pi} - \frac{1}{2\pi} \left[\ln (1+y^2) \right]_2^{2\pi}$$

$$= \frac{\ln 5}{2} \left(1 - \frac{1}{\pi}\right) + 2 \left[\tan^{-1} 2\pi - \tan^{-1} 2 \right] - \frac{1}{2\pi} \ln \frac{1+4\pi^2}{5}.$$