MA 105 : Calculus

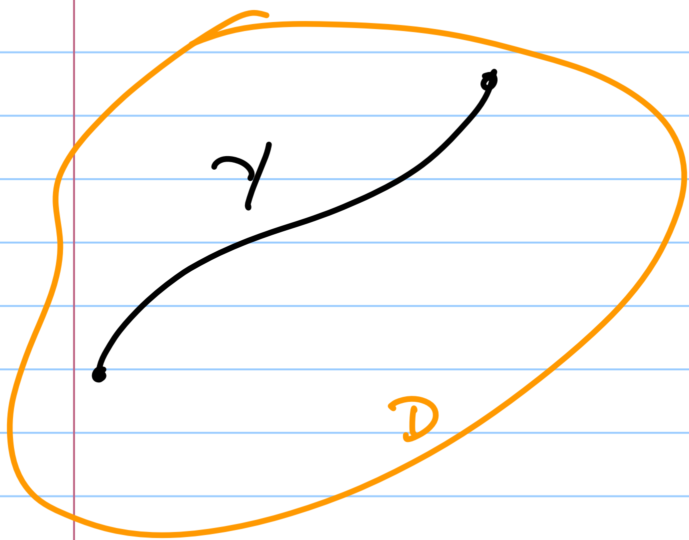
Sudhir R. Ghorpade

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai 400076, India
srg@math.iitb.ac.in
http://www.math.iitb.ac.in/∼srg/

IIT Goa

Autumn 2016

Line Integrals



Let $\gamma : [\alpha, \beta] \longrightarrow \mathbb{R}^n$ be a (piecewise smooth) path and $f : D \longrightarrow \mathbb{R}$ be a bounded function where $D \subseteq \mathbb{R}^n$ is such that "D contains γ", i.e., $\{\gamma(t) : t \in [\alpha, \beta]\} \subseteq D$.

The <u>line integral</u> of f over γ is denoted by

$\int_\gamma f \, ds$ or $\oint_\gamma f \, ds$ and is defined by

$$\int_\gamma f \, ds := \int_\alpha^\beta f(\gamma(t)) \, |\gamma'(t)| \, dt$$

provided the integral on right exists.

Note that if f is continuous, then $\int_\gamma f\,ds$ always exists.

The algebraic and order properties of Riemann integrals easily imply similar properties for line integrals of scalar fields.

The Case of Vector fields :

Suppose $\vec{F}$ is a vector field in $\mathbb{R}^n$ and $\gamma : [\alpha, \beta] \longrightarrow \mathbb{R}^n$ is a (piecewise smooth) path in $\mathbb{R}^n$. The line integral of $\vec{F}$ along γ is denoted by $\int_\gamma \vec{F} \cdot d\vec{s}$ and defined by $\int_\gamma \vec{F} \cdot d\vec{s} = \int_\alpha^\beta \vec{F}(\gamma(t)) \cdot \gamma'(t)\,dt$

For example if

$$\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$$

is a vector field in 3-space and

$$\gamma(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}, \quad t \in [\alpha, \beta]$$

a path contained in the domain of $\vec{F}$. Then

$$\int_\gamma \vec{F} \cdot d\vec{s} = \int_\alpha^\beta \vec{F}(x(t), y(t), z(t)) \cdot (x'(t), y'(t), z'(t))\,dt$$

The integral on the right can be written in various alternative ways as follows:

$$\int_\gamma \vec{F}\cdot d\vec{s} = \int_\gamma \vec{F}\cdot d\gamma \quad \text{or} \quad \int_\gamma \vec{F}\cdot d\vec{r}$$

($\vec{r}$ being the position at time t on the path γ, i.e., $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$)

$$= \int_\gamma \vec{F}\cdot\hat{T}\, ds$$

(recall $\hat{T} = \dfrac{d\gamma}{ds}$)

$$= \int_\alpha^\beta \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right) dt$$

$$= \int_\gamma P\,dx + Q\,dy + R\,dz\,.$$

Remark: When $\vec{F}$ represents a force, then $\vec{F}\cdot\hat{T}$ is the scalar component of $\vec{F}$ in the direction of curve's unit tangent and the line integral

$$\int_\gamma \vec{F}\cdot\hat{T}\,ds = \int_\gamma \vec{F}\cdot d\vec{s}$$

is the work done by the force $\vec{F}$ over the path γ.

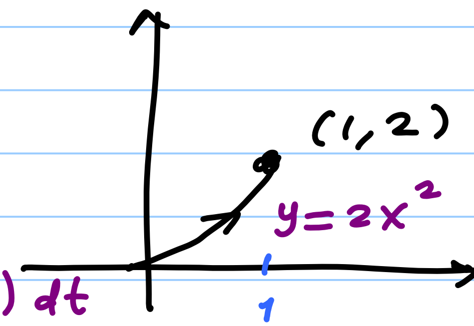
On the other hand when $\vec{F}$ represents the velocity field in a moving fluid, then

$$\int_\gamma \vec{F}\cdot\hat{T}\,ds$$

is the flow along the path γ or the circulation around γ in case γ is a closed curve

Example : Suppose C is the parabolic path in the plane from the origin to the point $P=(1,2)$ along the parabola $y=2x^2$ and

$$\vec{F} = xy\,\vec{i} + y^2\,\vec{j}$$

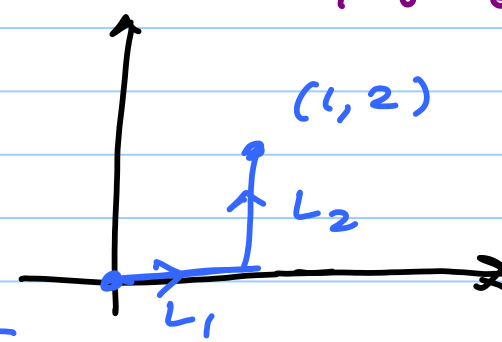


Then

$$\oint_C \vec{F}\cdot d\vec{s} = \int_0^1 \left(t(2t^2),\ 4t^4\right)\cdot(1,4t)\,dt$$

$$= \int_0^1 2t^3 + 16t^5\,dt = \frac{2}{4}+\frac{16}{6} = \frac{19}{6}$$

$\vec{r}(t) = (t,\ 2t^2)$ $\quad \vec{r}\,'(t) = (1,\ 4t)$



$$\oint_{L_1+L_2} \vec{F}\cdot d\vec{s} = \int_{L_1} + \int_{L_2} \vec{F}\cdot d\vec{s}$$

$$= 0 + \int_0 (t,\ t^2)\cdot(0,1)\,dt = \frac{8}{3}$$

Quick Recap of Line Integrals

Note Title 10/20/2011

Recall that we have discussed

Line integral of a scalar field:

$$\int_\gamma f\, ds := \int_\alpha^\beta f(\gamma(t))\, |\gamma'(t)|\, dt$$

Line integral of a vector field [Work integral or Flow integral]

$$\int_\gamma \vec{F} \cdot d\vec{s} := \int_\alpha^\beta \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt$$

where γ is a (piecewise smooth) path in n-space given by $\gamma : [\alpha, \beta] \to \mathbb{R}^n$ and f (resp. $\vec{F}$) is a scalar field (resp: vector field) in n-space whose domain contains $\gamma([\alpha, \beta])$.

We have noted that the line integral of a vector field $\vec{F}$ along γ can also be written as

$$\int_\gamma \vec{F} \cdot \hat{T} \, ds \quad \text{or} \quad \int_\gamma \vec{F} \cdot d\vec{r}$$

where $\hat{T} = \dfrac{d\gamma}{ds}$ denotes the unit tangent vector.

When, for example, $n = 3$, and $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, we could also write

$$\int_\gamma \vec{F} \cdot d\vec{s} \quad \text{as} \quad \int_\gamma P\,dx + Q\,dy + R\,dz .$$

Note that for a piecewise smooth path γ one often sets

$$ds = |\gamma'(t)|\,dt \quad \text{and} \quad d\vec{s} = \gamma'(t)\,dt$$

Examples : (1) [Tut Sheet No. 9, Q. 7]

$$f(x,y) = (x^2 - 2xy)\vec{i} + (y^2 - 2xy)\vec{j}$$

C : path from $(-1,1)$ to $(1,1)$ along $y = x^2$.

The line integral of the vector field f along C is

$$\int_C f \cdot d\vec{s} = \int_{-1}^{1} f(t, t^2) \cdot (\vec{i} + 2t\vec{j})\, dt$$

$$= \int_{-1}^{1} [(t^2 - 2t^3) + 2t(t^4 - 2t^3)]\, dt = \frac{14}{15}.$$

(2) [Tut 9, Q. 8] The line integral of the vector field $f(x,y) = (x^2+y^2)\vec{i} + (x-y)\vec{j}$ once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in the counterclockwise direction is $\int_C f \cdot d\vec{s} = \int_0^{2\pi} f(a\cos\theta, b\sin\theta) \cdot (-a\sin\theta\,\hat{i} + b\cos\theta\,\vec{j})\, d\theta$

$= \pi ab.$

Example [Tut Sheet No. 9, Q. 10]

C : intersection of two surfaces $z = xy$ and $x^2 + y^2 = 1$

Parametrically C is given by

$$x = \cos\theta, \quad y = \sin\theta, \quad z = \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta, \quad 0 \le \theta \le 2\pi$$

Thus

$$\oint_C y\,dx + z\,dy + x\,dz$$

$$= \int_0^{2\pi} \left[\sin\theta(-\sin\theta) + \frac{1}{2}\sin 2\theta(\cos\theta) + \cos\theta\cos 2\theta\right] d\theta$$

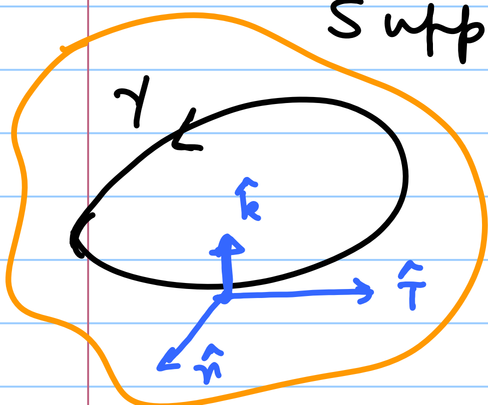
$$= -\pi.$$

We will now introduce another variant of a line integral that is quite useful in practice.

Flux integral

Suppose γ is a closed path in $\mathbb{R}^2$ and

$$\vec{F} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$$



is a continuous vector field in the plane and $\hat{n}$ is the outward unit normal to γ, then the line integral

$$\int_\gamma \vec{F}\cdot\hat{n}\,ds$$

is called the flux integral.

Its value is called the flux of $\vec{F}$ along γ (especially when $\vec{F}$ represents the velocity field of a fluid).

Remark: The outward unit normal $\hat{n}$ is given by

$$\hat{n} = \begin{cases} \hat{T} \times \hat{k} & \text{if } C \text{ is oriented counterclockwise,} \\ \hat{k} \times \hat{T} & \text{if } C \text{ is oriented clockwise.} \end{cases}$$

Since

$$\hat{T} = \frac{d\gamma}{ds} = \frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j}$$

we have for a closed curve, oriented counterclockwise,

$$\hat{n} = \hat{T} \times \vec{k} = \begin{vmatrix} i & j & k \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{dy}{ds}\vec{i} - \frac{dx}{ds}\vec{j}$$

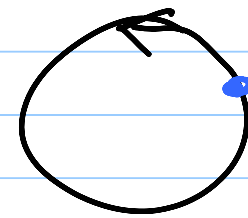
and thus

$$\int_\gamma \vec{F} \cdot \hat{n}\, ds = \int_\gamma \left(P\frac{dy}{ds} - Q\frac{dx}{ds}\right) ds = \int_\gamma P\,dy - Q\,dx.$$

Example:

Flux of $\vec{F} = \underbrace{(x-y)}_{P}\,\vec{i} + \underbrace{x}_{Q}\,\vec{j}$ along the unit circle C

$$= \int_C P\,dy - Q\,dx$$



$(\cos t, \sin t)$

$$= \int_0^{2\pi} \left[(\cos t - \sin t)(\cos t) - (\cos t)(-\sin t)\right] dt$$

$$= \int_0^{2\pi} \cos^2 t\, dt = \int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt = \pi.$$

Circulation of $\vec{F}$ along C

$$= \int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy = \int_0^{2\pi} (-\sin t \cos t + \sin^2 t + \cos^2 t)\, dt$$

$$= 2\pi.$$

Path Independence

If $\vec{F}$ is a vector field on an open subset D in n-space such that the value of

$$\int_{\gamma} \vec{F} \cdot d\vec{s}$$

depends only on the end-points of γ [i.e., for any $A, B \in D$ and a path γ from A to B, the integral $\int_{\gamma} \vec{F} \cdot d\vec{s}$ is independent of the choice of path in D from A to B], then the integral $\int \vec{F} \cdot d\vec{s}$ is said to be path independent in D.

Example: Suppose $\vec{F} = \nabla f$ for some smooth scalar field f and suppose γ is a smooth path. Then

$$\int_\gamma \vec{F} \cdot d\vec{s} = \int_\alpha^\beta \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt$$

Now if $h : [\alpha, \beta] \to \mathbb{R}$ is defined by

$$h(t) = f(\gamma(t))$$

then h is differentiable and by the Chain Rule,

$$h'(t) = \nabla f\,(\gamma(t)) \cdot \gamma'(t) = \vec{F}(\gamma(t)) \cdot \gamma'(t)$$

and thus

$$\int_\gamma \vec{F} \cdot d\vec{s} = \int_\alpha^\beta h'(t)\,dt = h(\beta) - h(\alpha) = f(B) - f(A)$$

where $A = \gamma(\alpha)$ and $B = \gamma(\beta)$ are the endpoints of γ.

Thus if $\vec{F} = \nabla f$ then $\int_\gamma \vec{F} \cdot d\vec{s}$ is path independent.

Question: If $\vec{F}$ is a continuous vector field on a domain D in n-space such that $\int_\gamma \vec{F} \cdot d\vec{s}$ is path-independent, then is $\vec{F}$ conservative, i.e., is $\vec{F} = \nabla f$ for some scalar field f? [Such a function f, if it exists, is called a potential function for $\vec{F}$.]

Answer: Yes, provided D is open & path-connected

Characterization of path-independence:

Let $\vec{F}$ be a continuous vector field on $D \subseteq \mathbb{R}^n$.
Then

$$\int_\gamma \vec{F} \cdot d\vec{s} \text{ is path-independent for any path } \gamma \text{ in } D$$

$$\Longleftrightarrow \int_C \vec{F} \cdot d\vec{s} = 0 \text{ for any closed path } C \text{ in } D.$$

Necessary condition for a conservative field:

Suppose $\vec{F} = P\vec{i} + Q\vec{j}$ is a smooth vector field in the plane. Then

$\vec{F}$ is conservative $\Longrightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} \Longleftrightarrow \operatorname{curl} \vec{F} = \vec{0}$.

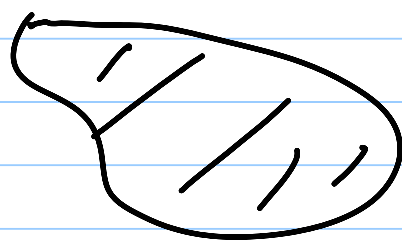
[Indeed, $\vec{F} = \nabla f \Longrightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial Q}{\partial x}$.]

Question: Suppose a vector field $\vec{F}$ is irrotational

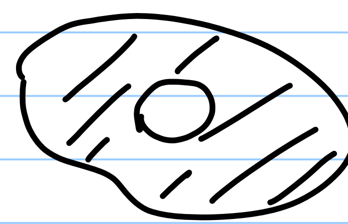
i.e., suppose $\operatorname{curl}(\vec{F}) = 0$. Then is $\vec{F}$ conservative?

i.e., does there exist a potential function f such that $\vec{F} = \nabla f$?

Answer: Yes, provided the domain D of $\vec{F}$ is "simply connected", i.e., if for every simple closed curve γ in D, the region enclosed by γ is in D.



simply connected



not simply connected

The condition that

$$\vec{F} = P\,\vec{i} + Q\,\vec{j}$$

is conservative (i.e., $\vec{F} = \nabla f$ for some f) is sometimes expressed by saying that the "differential form" $P\,dx + Q\,dy$ is exact

[$\vec{F} = \nabla f \Longrightarrow P = \dfrac{\partial f}{\partial x}$, $Q = \dfrac{\partial f}{\partial y}$

$$\Longrightarrow P\,dx + Q\,dy = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = df.$$]

Necessary condition for exactness in 3-space:

$\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$ is conservative, i.e., $P\,dx + Q\,dy + R\,dz$ is exact

$$\Longrightarrow \operatorname{curl}(\vec{F}) = 0, \text{ i.e., } \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \text{ and } \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

We have seen that

if $\vec{F}$ is conservative, i.e., if $\vec{F} = \nabla f$ for some smooth function f on D,

then $\int \vec{F} \cdot d\vec{s}$ is path-independent in D.

In fact,

$$\vec{F} = \nabla f \ \& \ \gamma \text{ path from } A \text{ to } B \Rightarrow \int_{\gamma} \vec{F} \cdot d\vec{s} = f(B) - f(A).$$

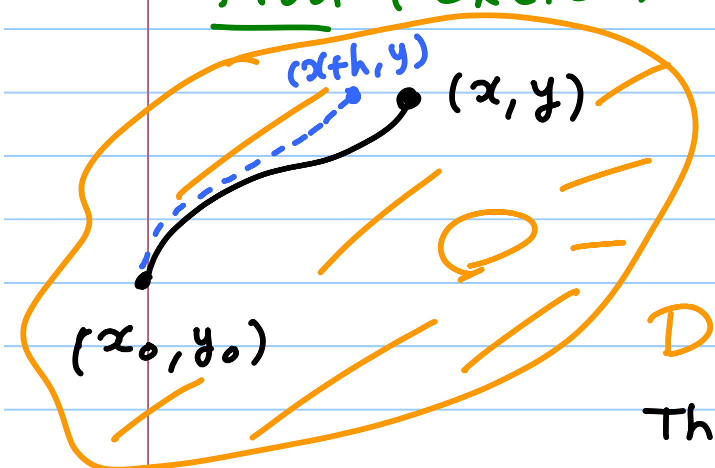
[Fundamental Theorem of Calculus for Line Integrals]

Fact: The converse is also true, provided D is an open, connected set.

More precisely, we have the following:

Theorem Let $D \subseteq \mathbb{R}^n$ be open & path-connected. If $\vec{F}$ is a continuous vector field on D such that $\int \vec{F} \cdot d\vec{s}$ is path-independent in D, then $\vec{F}$ is conservative, i.e., $\vec{F} = \nabla f$ for some smooth $f : D \to \mathbb{R}$.

Proof (Sketch) : Suppose, for simplicity, that $n = 2$.



Fix $P_0 = (x_0, y_0) \in D$. For any $P = (x, y)$ in D, define

$$f(x, y) := \int_\gamma \vec{F} \cdot d\vec{s}$$

where γ is any path from P_0 to P.

Then f is well-defined since $\int \vec{F} \cdot d\vec{s}$ is path-indep.

Moreover, for any $(x, y) \in D$ and $h \in \mathbb{R}$ with $|h|$ small, we can join $(x+h, y)$ & (x, y) by a straight line, say L. Thus,

$$f(x+h, y) - f(x,y) = \int_L \vec{F} \cdot d\vec{s}$$

$$= \int_0^1 \vec{F}(x+th,\, y) \cdot (h\vec{i})\, dt$$

and thus if $\vec{F} = F_1 \vec{i} + F_2 \vec{j}$, then for $h \neq 0$ with $|h|$ small

$$\frac{f(x+h, y) - f(x,y)}{h} = \int_0^1 F_1(x+th,\, y)\, dt$$

and consequently $\frac{\partial f}{\partial x}$ exists and is equal to $F_1(x,y)$.

Similarly $\frac{\partial f}{\partial y}$ exists and is $F_2(x,y)$. In other words,

f is smooth and $\vec{F} = \nabla f$, as desired.

We have also seen that

$\vec{F}$ is conservative in $D \subseteq \mathbb{R}^3 \Longrightarrow \operatorname{curl}(\vec{F}) = 0$.

Here the converse is true, provided the region

D is "simply connected".

Example [Tut Sheet No. 10, Q. 9]

Consider $\vec{F} = \dfrac{-y}{x^2+y^2}\vec{i} + \dfrac{x}{x^2+y^2}\vec{j} = F_1(x,y)\,\vec{i} + F_2(x,y)\,\vec{j}$

We have

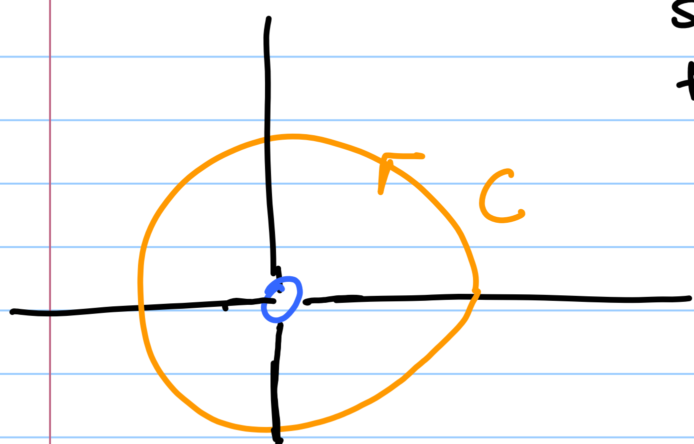
$$\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = 0\,\vec{i} + 0\,\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

$$= \vec{0}$$

$\left[\text{since } \dfrac{\partial F_1}{\partial y} = \dfrac{-x^2+y^2}{(x^2+y^2)^2} = \dfrac{\partial F_2}{\partial x}\right]$

However $\vec{F} \neq \nabla f$ for any smooth scalar field f on $D := \mathbb{R}^2 \setminus \{(0,0)\}$ [or $D = \mathbb{R}^3 \setminus \{(0,0,z) : z \in \mathbb{R}\}$]

To see this, it suffices to note that if C is the standard unit circle [in the xy-plane] then



$$\int_C \vec{F} \cdot d\vec{s}$$

$$= \int_0^{2\pi} (-\sin\theta\,\vec{i} + \cos\theta\,\vec{j}) \cdot (-\sin\theta\,\vec{i} + \cos\theta\,\vec{j})\,d\theta$$

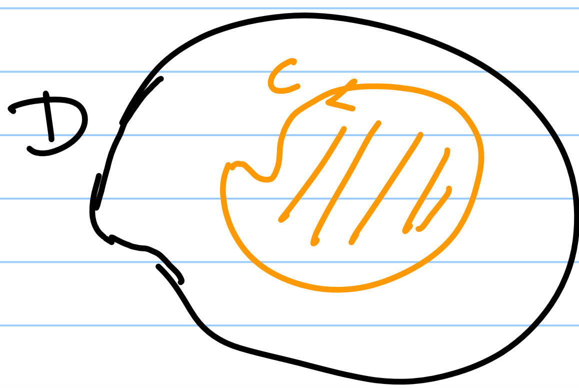
$$= 2\pi \neq 0$$

[Exer: Calculate $\int_\gamma \vec{F} \cdot ds$ if γ is the circle of radius 1 centered at $(2,0)$.]

The difficulty in the previous example is that the domain D has a "hole" or more precisely

D is not simply connected.

[A path-connected subset D of $\mathbb{R}^n$ is said to be <u>simply connected</u> if for every simple closed curve C in D, the region "enclosed by C" lies completely in D.]



Examples of simply-connected domains:
$\mathbb{R}^n$, open convex sets in $\mathbb{R}^n$, closed discs, rectangular regions.

If $D \subseteq \mathbb{R}^2$, then D is simply-conn. $\iff$ both D & $\mathbb{R}^2 \setminus D$ are path-connected.]