MA 105 : Calculus

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Autumn 2016

Green's Theorem

This is an important result that relates line integrals to double integrals and it can be viewed as a 2-dimensional analogue of the Fundamental Theorem of Calculus.

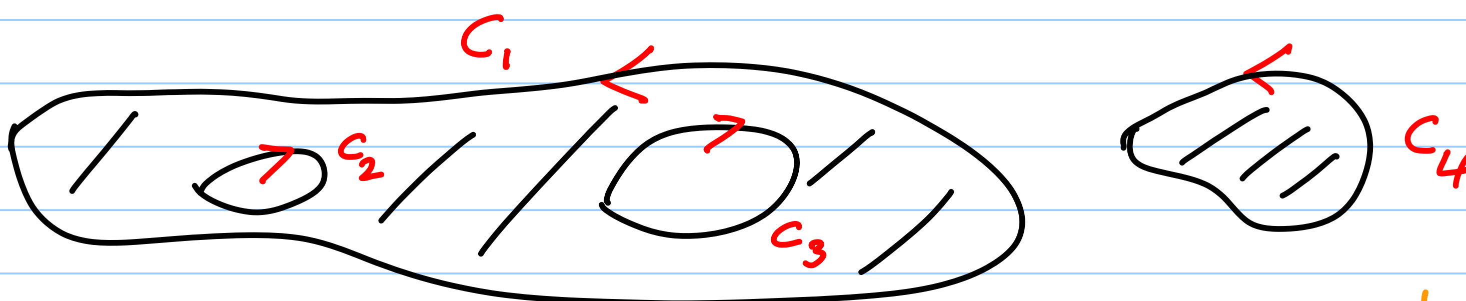
Here is a formal statement of Green's theorem in one of its simplest forms.

Theorem: Let D be a region in $\mathbb{R}^2$ such that the boundary of D is a simple closed curve C (oriented counterclockwise). If $\vec{F} = P\vec{i} + Q\vec{j}$ is a smooth vector field on an open subset of $\mathbb{R}^2$ containing $D \cup C$, then

$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d(x,y).}$$

Remark: Green's theorem holds more generally when D is a region in $\mathbb{R}^2$ whose boundary, denoted ∂D consists of a finite number of simple closed curves $C_1, \dots, C_m$ provided ∂D is positively oriented. In this case

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d(x,y).$$



positively oriented $\longleftrightarrow$ D is on your left as you travel along ∂D.

Alternative Formulations of Green's Theorem

Hypothesis being the same, the key formula in Green's theorem, namely,

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d(x,y)$$

can be expressed as follows:

- $$\int_{\partial D} \vec{F}\cdot d\vec{s} = \int_{\partial D} \vec{F}\cdot\vec{T}\,ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d(x,y) = \iint_D (\operatorname{curl}\vec{F})\cdot\vec{k}\; d(x,y)$$

circulation

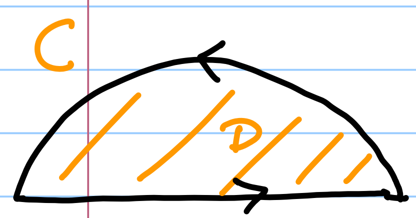
[Circulation-Curl Form or Tangential Form]

- $$\int_{\partial D} \vec{F}\cdot\vec{n}\,ds = \int_{\partial D} P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) d(x,y) = \iint_D \operatorname{div}\vec{F}\, d(x,y)$$

outward flux

[Flux-Divergence Form or Normal Form]

Examples: Compute $\oint_C y^2\,dx + 3xy\,dy$, where C is the counterclockwise boundary of the upper-half unit disc.



By Green's theorem

$$\oint_C y^2dx+3xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\right] d(x,y)$$

where D is the upper-half unit disc

$$D = \{(x,y)\in \mathbb{R}^2 : -1 \leq x \leq 1,\ 0 \leq y \leq \sqrt{1-x^2}\}.$$

Thus

$$\oint_C y^2dx+3xy\,dy = \int_{-1}^{1}\left(\int_0^{\sqrt{1-x^2}} y\,dy\right)dx = \int_{-1}^{1}\frac{1-x^2}{2}\,dx = \frac{2}{3}.$$

[**Exercise:** Compute the line integral directly and verify that you get the same answer! Which method is easier?]

Applications of Green's Theorem

- **Calculation of areas :**

We have seen that for a planar region D

$$\text{Area}(D) = \iint_D 1 \, d(x,y) = \int_{\partial D} P\,dx + Q\,dy$$

where P, Q are nice functions such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$.

For example

$$P = -\frac{y}{2}, \quad Q = \frac{x}{2} \quad \rightsquigarrow \quad \text{Area}(D) = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx$$

$$P = 0, \quad Q = x \quad \rightsquigarrow \quad \text{Area}(D) = \int_{\partial D} x\,dy.$$

$$P = -y, \quad Q = 0 \quad \rightsquigarrow \quad \text{Area}(D) = -\int_{\partial D} y\,dx.$$

Thus if ∂D is a simple closed curve C given parametrically by $x = x(t)$, $y = y(t)$, $t \in [\alpha, \beta]$, then

$$\text{Area}(D) = \frac{1}{2}\int_\alpha^\beta [x(t)\,y'(t) - x'(t)\,y(t)]\,dt = \frac{1}{2}\int_\alpha^\beta \underbrace{\begin{vmatrix} x & y \\ x' & y' \end{vmatrix}}_{W(x,\,y)} dt.$$

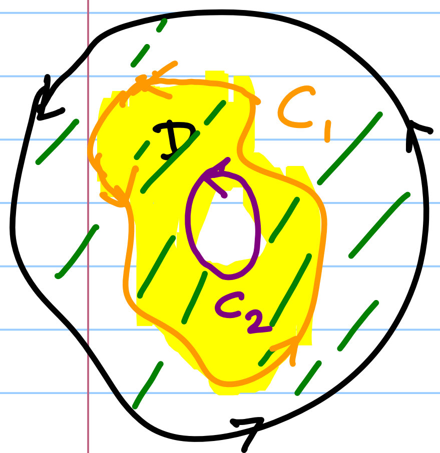
the Wronskian of $x(t), y(t)$

As a special case if C is given by a polar equation $r = r(\theta)$, $\theta \in [\alpha, \beta]$, then we may take $x(\theta) = r(\theta)\cos\theta$, $y(\theta) = r(\theta)\sin\theta$, $\theta \in [\alpha, \beta]$ to obtain

$$\text{Area}(D) = \frac{1}{2}\int_\alpha^\beta \begin{vmatrix} r(\theta)\cos\theta & r(\theta)\sin\theta \\ r'(\theta)\cos\theta - r(\theta)\sin\theta & r'(\theta)\sin\theta + r(\theta)\cos\theta \end{vmatrix} d\theta$$

$$= \frac{1}{2}\int_\alpha^\beta r^2(\theta)\,d\theta.$$

• Invariance of certain line integrals



Suppose C_1, C_2 are simple closed curves such that C_2 lies in the interior of C_1 and suppose P, Q are smooth functions on an open set containing C_1, C_2 and the region between the interior of C_1 and exterior of C_2 and such that

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \quad \text{on this open set}$$

Then

$$\oint_{C_1} P\,dx + Q\,dy = \oint_{C_2} P\,dx + Q\,dy$$

provided C_1 and C_2 have the same orientation.

Proof: Apply Green's theorem to the region D bounded by C_1 and C_2. Note that $\partial D = C_1 - C_2$.

Proof of Green's Theorem (when D is an elementary region of type I as well as type II)

Suppose

$$D = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } \varphi_1(x) \leq y \leq \varphi_2(x)\}$$

for some continuous functions $\varphi_1, \varphi_2 : [a,b] \longrightarrow \mathbb{R}$ such that $\varphi_1 \leq \varphi_2$ and also

$$D = \{(x,y) \in \mathbb{R}^2 : c \leq y \leq d \text{ and } \psi_1(y) \leq x \leq \psi_2(y)\}$$

for some continuous functions $\psi_1, \psi_2 : [c,d] \longrightarrow \mathbb{R}$ such that $\psi_1 \leq \psi_2$. [Examples of such regions include disks and rectangles]

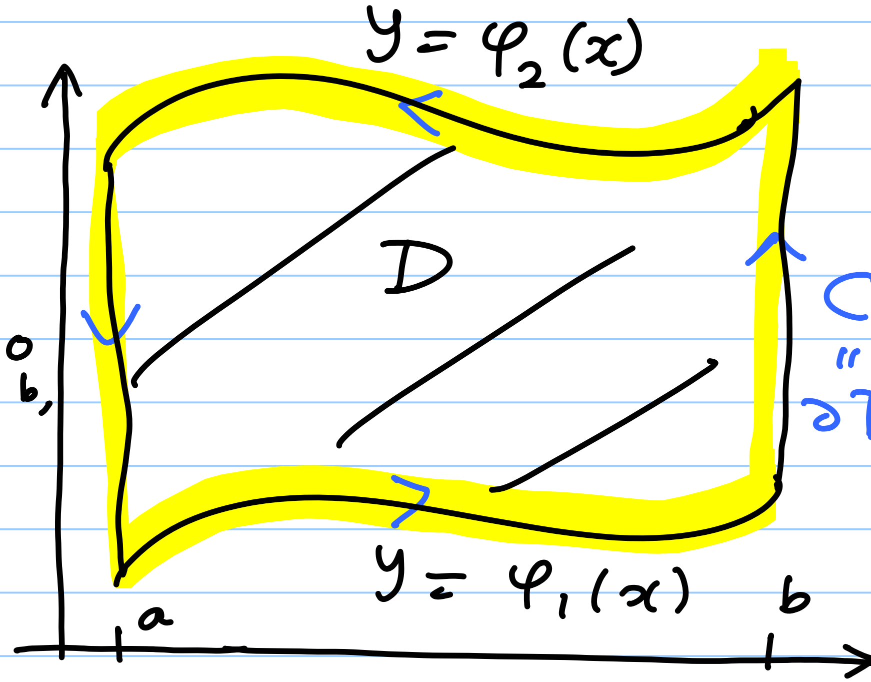
To show:
$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d(x,y).$$

First, since $D = \{(x,y) : a \le x \le b \ \& \ \varphi_1(x) \le y \le \varphi_2(x)\}$, by Fubini's theorem and the Funda. Theorem of Calculus,

$$\iint_D \frac{\partial P}{\partial y}\, d(x,y)$$

$$= \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}\, dy \right) dx$$

$$= \int_a^b \left[P(x, \varphi_2(x)) - P(x, \varphi_1(x)) \right] dx$$



On the other hand, since $dx = 0$ when $x = a$ or b,

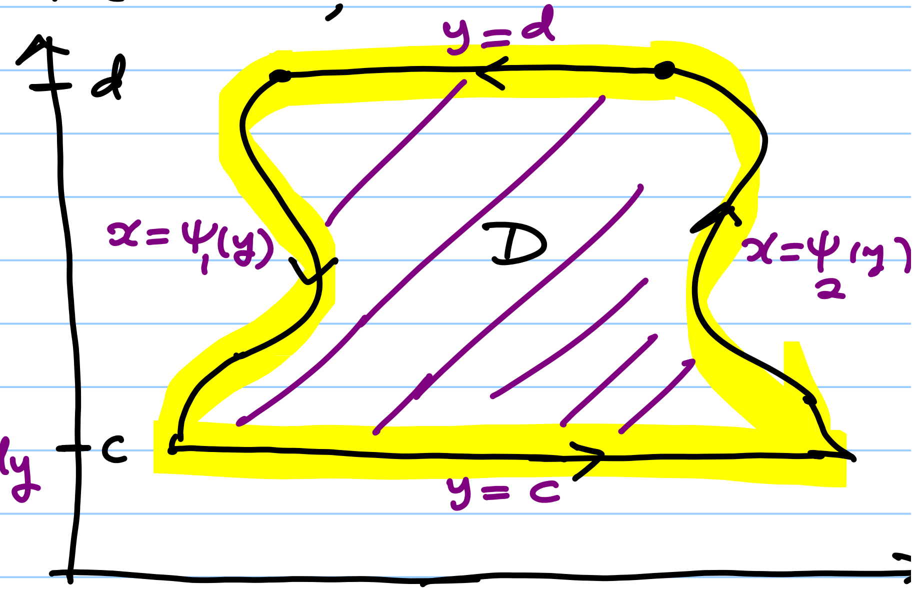
$$\oint_{\partial D} P\, dx = \int_a^b P(x, \varphi_1(x))\, dx$$

$$+ 0 + \int_b^a P(x, \varphi_2(x))\, dx + 0$$

$$= \int_a^b \left[P(x, \varphi_1(x)) - P(x, \varphi_2(x)) \right] dx$$

This shows that $\displaystyle\oint_{\partial D} P\, dx = - \iint_D \frac{\partial P}{\partial y}\, d(x,y).$

Next, since $D = \{ (x,y) \in \mathbb{R}^2 : c \leq y \leq d \ \& \ \psi_1(y) \leq x \leq \psi_2(y) \}$, by Fubini's Theorem and the FTC,



$$\iint_D \frac{\partial Q}{\partial x} \, d(x,y)$$

$$= \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial Q}{\partial x} \, dx \right) dy$$

$$= \int_c^d \left[Q(\psi_2(y), y) - Q(\psi_1(y), y) \right] dy$$

On the other hand,

$$\oint_{\partial D} Q \, dy = \int_d^c Q(\psi_1(y), y) + 0 + \int_c^d Q(\psi_2(y), y) \, dy + 0$$

$$= \int_c^d \left[Q(\psi_2(y), y) - Q(\psi_1(y), y) \right] dy = \iint_D \frac{\partial Q}{\partial x} \, d(x,y).$$

Combining, we obtain, $\displaystyle \oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d(x,y)$, as desired.

We will now move up one dimension and pass on from paths and line integrals to

Surfaces and Surface Integrals

Just as a curve in the plane may be given implicitly [by an equation of the form $f(x,y)=0$] or parametrically [by parametric equations $x=x(t)$, $y=y(t)$] a surface in 3-space is given either by an

implicit equation : $F(x,y,z)=0$, $(x,y,z)\in D$ $[D\subseteq\mathbb{R}^3]$

or a parametric equation which will now involve two indepent parameters, say u and v :

$$\Phi(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}, \quad (u,v)\in E,\ E\subseteq\mathbb{R}^2.$$

We will refer to the corresponding map

$$\Phi : E \longrightarrow \mathbb{R}^3$$
$$(u,v) \longmapsto (x(u,v),\ y(u,v),\ z(u,v))$$

as a <u>parametrized surface</u> and it will be generally assumed that Φ is smooth, i.e., the functions

$$x, y, z : E \to \mathbb{R}$$

extend to an open set $U \subseteq \mathbb{R}^2$ containing E and have continuous partial derivatives on U. We will also tacitly assume that the domain E is a nice set in the sense that E has an area, i.e. $\iint_E 1 \, d(x,y)$ exists [quite often, E would be a rectangle]

Example: (1) The graph of any function

$$f : E \longrightarrow \mathbb{R}, \qquad E \subseteq \mathbb{R}^2$$

is a surface in $\mathbb{R}^3$ given implicitly by

$$F(x,y,z) := z - f(x,y) = 0 \quad , \text{i.e}, \quad z = f(x,y)$$

It can also be viewed as a parametric surface given by

$$\Phi(u,v) = u\,\hat{i} + v\,\hat{j} + f(u,v)\,\hat{k} \quad , (u,v) \in E.$$

(2) Cylinder given, for example, by $x^2+y^2=a^2$ in $\mathbb{R}^3$ whose portion from $z=0$ to $z=h$ is parametrically given by

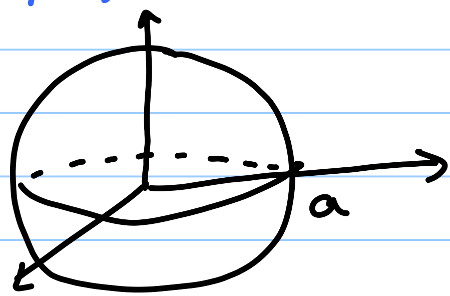
$$\Phi(u,v) = a\cos u\,\hat{i} + a\sin u\,\hat{j} + v\,\hat{k},$$

$$(u,v) \in [0, 2\pi] \times [0, h].$$

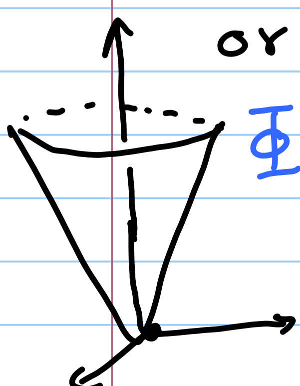
③ The sphere given by $x^2+y^2+z^2=a^2$ or by

$$\Phi(u,v) = a\sin u\cos v\,\hat{i} + a\sin u\sin v\,\hat{j} + a\cos u\,\hat{k},$$

$$(u,v)\in E = [0,\pi]\times[0,2\pi].$$



④ The cone given by $z^2 = x^2+y^2$, $0\le z\le h$



or by

$$\Phi(u,v) = v\cos u\,\hat{i} + v\sin u\,\hat{j} + v\,\hat{k},$$

$$(u,v)\in[0,2\pi]\times[0,h]$$

[Note: Thomas & Finney often uses the notation $\vec{r}(u,v)$ instead of $\Phi(u,v)$]

Normal vector to surfaces

Suppose $\Phi : E \to \mathbb{R}^3$ is a parametrized surface and C is a curve on this surface passing through $P = (x_0, y_0, z_0) = \Phi(u_0, v_0)$ for some $(u_0, v_0) \in E$. We may suppose C is given by

$$\gamma(t) = \Phi(u(t), v(t)) \quad , \quad t \in [\alpha, \beta]$$

where $u, v : [\alpha, \beta] \to \mathbb{R}$ are such that $(u(t), v(t)) \in E$ $\forall t$

and $(u_0, v_0) = (u(t_0), v(t_0))$ for some $t_0 \in (\alpha, \beta)$.

Now by Chain Rule, the tangent vector to C at P is given by

$$\gamma'(t_0) = \Phi_u(u_0, v_0)\, u'(t_0) + \Phi_v(u_0, v_0)\, v'(t_0) \quad .$$

It follows that the tangent vector $\gamma'(t_0)$ is always orthogonal to

$$\Phi_u(u_0, v_0) \times \Phi_v(u_0, v_0)$$

With this in view, we define

$$(\Phi_u \times \Phi_v)(u_0, v_0)$$

to be a <u>normal vector</u> to Φ, provided it is nonzero. In this case

$$\hat{n} = \frac{\Phi_u \times \Phi_v (u_0, v_0)}{|\Phi_u \times \Phi_v (u_0, v_0)|}$$

is called the unit normal vector to Φ at P.

Def : A (smooth) parametrized surface given by Φ is said to be <u>regular</u> if the unit normal vector is defined at each point, i.e., if

$$(\Phi_u \times \Phi_v)(u_0, v_0) \neq 0 \quad \forall (u_0, v_0) \in E.$$

The plane

$$\Phi_u \times \Phi_v (u_0, v_0) \cdot \left[(x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k} \right] = 0$$

i.e.

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

where $(A, B, C) = (\Phi_u \times \Phi_v)(u_0, v_0)$

is called the <u>tangent plane</u> to Φ at P.