MA 105 : Calculus

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Consequences of the Critical Point Lemma

- <u>IVP for derivatives</u>

  I an interval, $f : I \to \mathbb{R}$ differentiable
  $\Longrightarrow$ f' has the IVP on I

<u>Proof (sketch):</u> Let $a, b \in I$ with $a < b$ and $r \in \mathbb{R}$ be between $f'(a)$ & $f'(b)$. First, suppose $f'(a) < r < f'(b)$. Consider $g : [a, b] \longrightarrow \mathbb{R}$ defined by $g(x) := f(x) - rx$ for $x \in [a, b]$. Then g is differentiable and

$$\lim_{x \to a^+} \frac{g(x) - g(a)}{x - a} = g'(a) = f'(a) - r < 0$$

$\Longrightarrow$ $\exists\ \delta > 0$ s.t.

$$\frac{g(x) - g(a)}{x - a} < 0 \quad \forall\ x \in (a, a+\delta)$$

$\Longrightarrow$ $g(x) < g(a) \quad \forall\ x \in (a, a+\delta)$

$\Longrightarrow$ g can't attain its minimum at a.

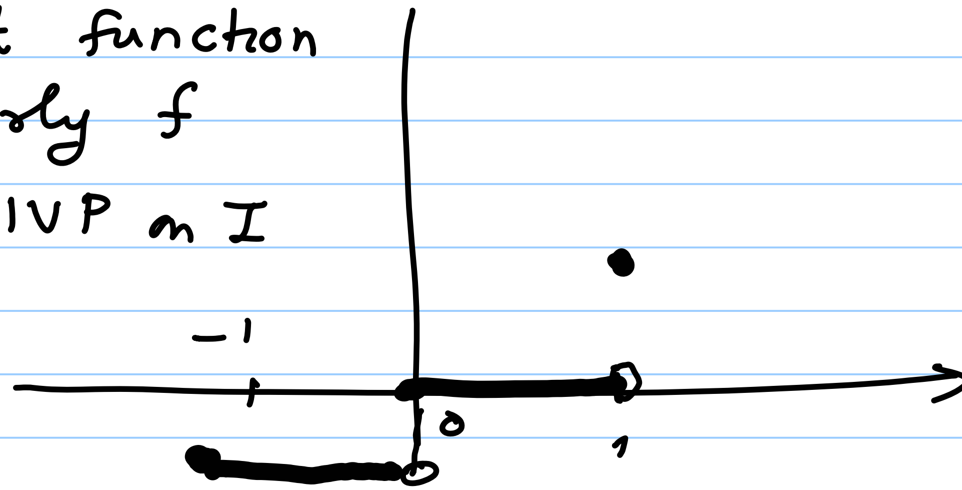
Similarly

$g'(b) = f'(b) - r > 0 \Longrightarrow g$ can't attain its min at b.

So g attains its minimum at some $c \in (a,b)$ and by CPL, $g'(c) = 0$, i.e., $f'(c) = r$. The case $f'(a) > f'(b)$ is similarly proved.

Remark: The IVP for derivatives can be used to give examples of functions f that have no antiderivative, i.e., $f \neq g'$ for any differentiable function g.

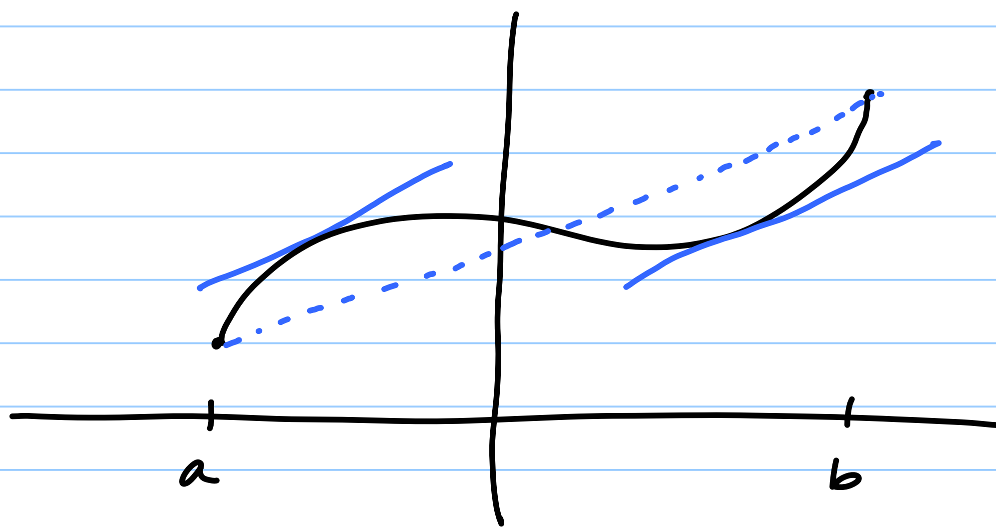
For example, let $I = [-1, 1]$ and $f: I \to \mathbb{R}$ be the integer part function $f(x) = [x]$. Clearly f does not have the IVP on I so $f \neq g'$ for any diff. $g: I \to \mathbb{R}$.



(Lagrange's) Mean Value Theorem [MVT]

$f : [a,b] \to \mathbb{R}$ continuous on $[a,b]$
and differentiable on (a,b)

$\implies f(b) - f(a) = f'(c)(b-a)$ for some $c \in (a,b)$.



some tangent is parallel to the chord joining $(a, f(a))$ & $(b, f(b))$.

Proof: Consider $F : [a, b] \longrightarrow \mathbb{R}$ defined by

$$F(x) := f(x) - f(a) - s(x-a)$$

where $s \in \mathbb{R}$ is so chosen that $F(a) = 0 = F(b)$, namely

$$s = \frac{f(b) - f(a)}{b - a}$$

Now Rolle's Theorem applies to $F : [a,b] \rightarrow \mathbb{R}$. So $F'(c) = 0$ for some $c \in (a,b)$, which implies $s = f'(c)$ so that $f(b) - f(a) = f'(c)(b-a)$ for some $c \in (a, b)$.

Consequences of MVT:

① I interval (containing more than one point),
$f : I \to \mathbb{R}$ diff. and $f'(x) = 0 \ \forall x \in I$
$\Longrightarrow$ f is a constant function on I.

② $f : [a, b] \to \mathbb{R}$ cont. on $[a, b]$, diff. on (a, b)
If $m, M \in \mathbb{R}$ are s.t. $m \leq f'(x) \leq M \ \forall x \in (a, b)$
then
$$m(b-a) \leq f(b) - f(a) \leq M(b-a)$$
[Mean Value Inequality]

Example (Approximation of Square Root)

$$f(x) = \sqrt{x}, \quad x \in [m, m+1] \qquad (m \in \mathbb{N})$$

$$\frac{1}{2\sqrt{m+1}} < \sqrt{m+1} - \sqrt{m} < \frac{1}{2\sqrt{m}}$$

e.g., $m = 1$ gives

$$1 + \frac{1}{2\sqrt{2}} < \sqrt{2} < 1 + \frac{1}{2} \implies \frac{4}{3} < \sqrt{2} < \frac{3}{2}$$

(Try to find similar approximations for $\sqrt{3}$, $\sqrt{5}$.)

Taylor's Theorem

Let $n \geq 0$, $f:[a,b] \to \mathbb{R}$ be such that

$f', f'', \ldots, f^{(n)}$ exist on $[a,b]$

and also

$f^{(n)}$ is continuous on $[a,b]$ and differentiable on (a,b).

Then

$\exists\ c \in (a,b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n$$

where $R_n = \dfrac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$.

Examples

- $n=0$: $$f(b) = f(a) + f'(c)(b-a) \quad \text{[MVT]}$$
- $n=1$: $$f(b) = f(a) + f'(a)(b-a) + \frac{f''(c)}{2}(b-a)^2$$

[Extended MVT]

and so on.

A Variant of Taylor's Theorem

$f : [a,b] \to \mathbb{R}$ as in Taylor's Theorem

$\Rightarrow \exists\, c \in (a,b)$ s.t.

$$f(a) = f(b) + f'(b)(a-b) + \frac{f''(b)}{2}(a-b)^2 + \cdots + \frac{f^{(n)}(b)(a-b)^n}{n!} + \frac{f^{(n+1)}(c)}{(n+1)!}(a-b)^{n+1}.$$

Proof of Taylor's Theorem

For $x \in [a, b]$, let

$$P(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

To show: $\exists\, c \in (a, b)$ such that

$$f(b) = P(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

So consider $F : [a, b] \longrightarrow \mathbb{R}$ defined by

$$F(x) = f(x) - P(x) - s\,(x-a)^{n+1}$$ where

$$s = [f(b) - P(b)] / (b-a)^{n+1}.$$

Thus we want to show: $\boxed{S = \dfrac{f^{(n+1)}(c)}{(n+1)!} \text{ for some } c \in (a,b).}$

Observe that

$$F(a) = 0 = F(b)$$

$$\Longrightarrow F'(c_1) = 0 \quad \text{for some } c_1 \in (a,b) \quad \text{[by Rolle's Thm]}$$

Now

$$F'(a) = f'(a) - P'(a) = 0 = F'(c_1)$$

$$\Longrightarrow F''(c_2) = 0 \quad \text{for some } c_2 \in (a, c_1) \quad \text{[by Rolle's Theorem]}$$

Next

$$F''(a) = F''(c_2) = 0 \Longrightarrow F'''(c_3) = 0 \text{ for some } c_3 \in (a, c_2)$$

and so on

Continuing in this way, we find

$$\exists \ c = c_{n+1} \in (a, c_n) \subseteq (a,b) \ \text{ s.t. }$$

$$F^{(n+1)}(c) = 0$$

$$\Rightarrow f^{(n+1)}(c) = (n+1)! \, S$$

This proves the theorem.

Proof of the Variant of Taylor's Thm (with $a \leftrightarrow b$):

Apply Taylor's Theorem to $g : [a,b] \to \mathbb{R}$ defined by $\quad g(x) = f(a+b-x) \quad$ for $x \in [a,b]$.

Combined Statement of Taylor's Theorem

I an interval in $\mathbb{R}$, $a \in \mathbb{R}$.

$f: I \to \mathbb{R}$ s.t. $f', f'', \dots, f^{(n)}$ exist and are continuous on I and $f^{(n+1)}$ exists at every interior point of I

$\Longrightarrow$ for any $x \in I$, $x \neq a$, we have

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^{n}}_{P_n(x),\ n^{th}\text{ Taylor poly. of } f \text{ around } a} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{R_n(x)\text{ - remainder}}$$

Examples :

① $f(x) = \sin x$

We have

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \ \dots$$

$$f^{(k)}(x) = \begin{cases} (-1)^{k/2} \sin x & \text{if } k \text{ even}, \\ (-1)^{(k-1)/2} \cos x & \text{if } k \text{ odd}. \end{cases}$$

So the n^{th} Taylor polynomial of f around 0 is

$$P_n(x) = \begin{cases} x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots + (-1)^{(n-1)/2} \dfrac{x^n}{n!} & \text{if } n \text{ odd}, \\ x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots + (-1)^{(n-2)/2} \dfrac{x^{n-1}}{(n-1)!} & \text{if } n \text{ even}. \end{cases}$$

Note that by Taylor's Theorem

$$R_n(x) = \sin x - P_n(x)$$

$$= \begin{cases} (-1)^{n/2} \dfrac{(\cos c)\, x^{n+1}}{(n+1)!} & \text{if } n \text{ even}, \\ (-1)^{(n+1)/2} \dfrac{(\sin c)\, x^{n+1}}{(n+1)!} & \text{if } n \text{ odd}. \end{cases}$$

for some c bet^n 0 & x

So in any case

$$|R_n(x)| \leq \frac{x^{n+1}}{(n+1)!} \longrightarrow 0 \text{ as } n \to \infty$$

This shows that $P_n(x) \longrightarrow \sin x$ as $n \to \infty$ and so we may write

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdot$$

$$= \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n+1}}{(2n+1)!}$$

and refer to the RHS as the Taylor series of $f(x) = \sin x$ around 0.

2 In a similar manner, one can see that

$$\cos x = \sum_{n=0}^{\infty} (-1)^{n} \frac{x^{2n}}{(2n)!}$$

Remark: Taylor series around 0 is sometimes called Maclaurin series.