MA 105 : Calculus

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APPLICATIONS

- **Monotonicity, Convexity and Concavity**

Let I be an interval and $f: I \to \mathbb{R}$ be any function. We say that f is

- (monotonically) increasing on I if

$$x_1, x_2 \in I, \quad x_1 < x_2 \implies f(x_1) \leq f(x_2)$$

- (monotonically) decreasing on I if

$$x_1, x_2 \in I, \quad x_1 < x_2 \implies f(x_1) \geq f(x_2)$$

- strictly increasing on I if

$$x_1, x_2 \in I, \quad x_1 < x_2 \implies f(x_1) < f(x_2).$$

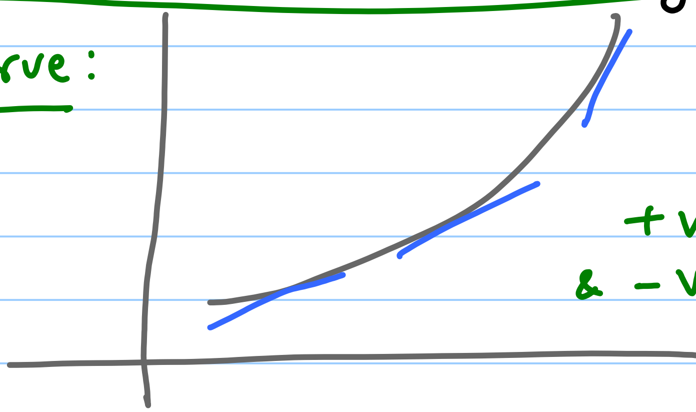
- strictly decreasing on I if

$$x_1, x_2 \in I, \quad x_1 < x_2 \implies f(x_1) > f(x_2).$$

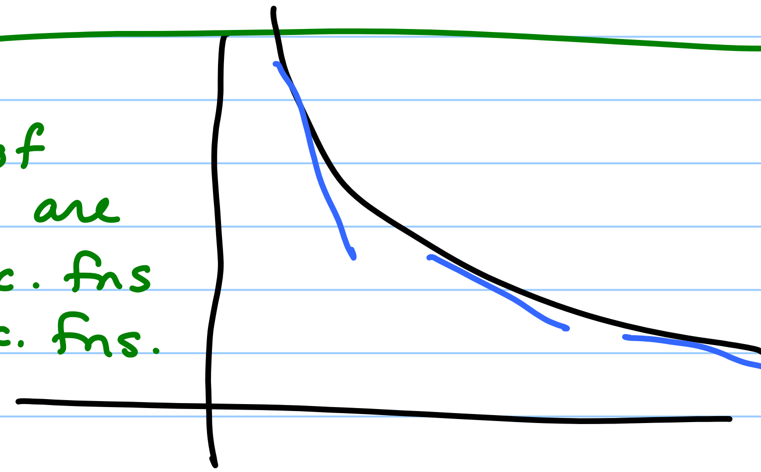
One says that f is

- monotonic (or monotone) on I if it is increasing or decreasing on I
- strictly monotonic (or strictly monotone) on I if it is strictly increasing or strictly decreasing on I.

Observe:



Slopes of tangents are +ve for inc. fns & −ve for dec. fns.



Derivatives and Monotonicity:

Prop. Let I be an interval (containing more than one point) and $f : I \to \mathbb{R}$ be differentiable. THEN:

i) $f' \geq 0$ on $I \iff f$ is increasing on I

ii) $f' \leq 0$ on $I \iff f$ is decreasing on I

iii) $f' > 0$ on $I \implies f$ is strictly increasing on I

iv) $f' < 0$ on $I \implies f$ is strictly decreasing on I.

Proof: Follows from the MVT!

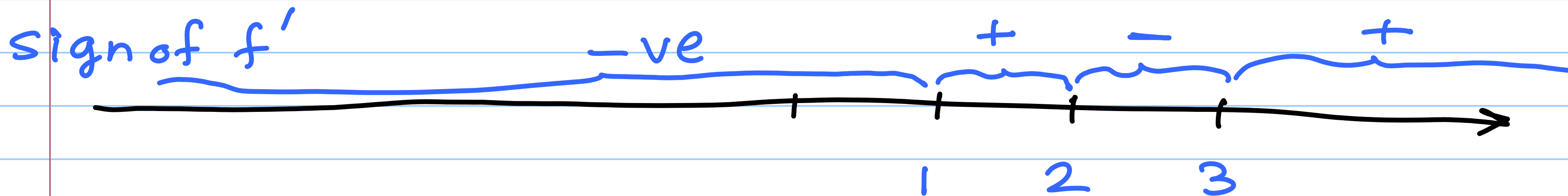
Example : The converse of iii) & iv) is not true, in general. For example, consider $f : [-1, 1] \to \mathbb{R}$ defined by

$f(x) = x^3$ or by $f(x) = -x^3$.

Example: $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 7$

$$f'(x) = 4x^3 - 24x^2 + 44x - 24$$

$$= 4(x-1)(x-2)(x-3)$$

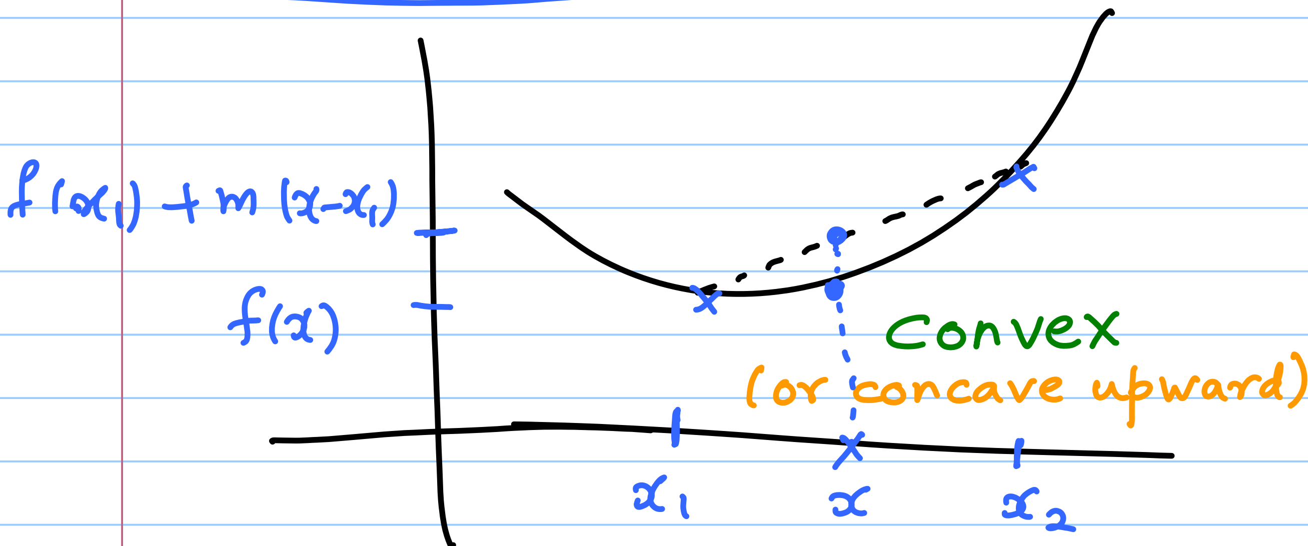


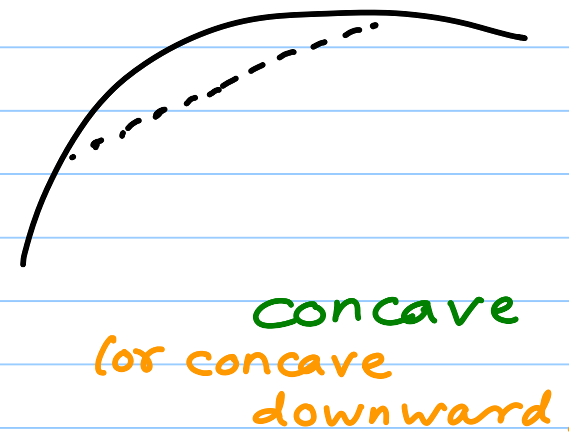
So

f is strictly decreasing on $(-\infty, +1)$ & $(2,3)$

and strictly increasing on $(1,2)$ & $(3,\infty)$.

Convexity & Concavity





Def: $f : I \to \mathbb{R}$ is said to be

- convex (or concave upward) on I if

$x_1, x_2, x \in I$, $x_1 < x < x_2 \implies f(x) \leq f(x_1) + \underbrace{\frac{f(x_2) - f(x_1)}{x_2 - x_1}}_{m} (x - x_1)$

- concave ..... ----- $\geq$ .....

Alternative Defn :

- $f : I \to \mathbb{R}$ is <u>convex</u> on I if

$$f(tx_1 + (1-t)x_2) \leq t f(x_1) + (1-t) f(x_2)$$

$\forall\, x_1, x_2 \in I \;\&\; t \in (0,1)$.

- $f : I \to \mathbb{R}$ is <u>concave</u> on I if

$$f(tx_1 + (1-t)x_2) \geq t f(x_1) + (1-t) f(x_2).$$

$\forall\, x_1, x_2 \in I \;\&\; t \in (0,1)$

<u>Example</u>: i) $f(x) = |x|$ is convex on $\mathbb{R}$.

ii) $f(x) = x^3$ is convex on $[0, \infty)$ & concave on $(-\infty, 0]$.

Relation with differentiability

Prop: Let I be an interval & $f: I \to \mathbb{R}$ be a differentiable function. Then

i) f' increasing on $I \iff f$ convex on I

ii) f' decreasing on $I \iff f$ concave on I

Proof: Skipped. [See [GL-1], p. 128 if you are interested.]

Corollary: Let I be an interval & $f: I \to \mathbb{R}$ be twice differentiable. Then:

i) $f'' \geq 0$ on $I \iff f$ convex on I

ii) $f'' \leq 0$ on $I \iff f$ concave on I.