Lecture 19

Note Title 10/10/2011

Recall that we have discussed

- double integrals of bounded real-valued functions on rectangles ($R = [a,b] \times [c,d]$)
  (defn. via (Riemann) double sums)

- basic properties, including Fubini's Theorem on rectangles

- extension to bounded functions $f : D \to \mathbb{R}$ on arbitrary bounded domains $D \subseteq \mathbb{R}^2$
  (defined via extending f to $f^* : R \to \mathbb{R}$, where R is a rectangle containing D and $f^*(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D, \\ 0 & \text{if } (x,y) \in R \setminus D. \end{cases}$)

It is readily seen that the algebraic and order properties of double integrals extend easily to the case of arbitrary bounded domains $D \subseteq \mathbb{R}^2$ instead of rectangles. Moreover, we have seen that we have a very useful result: Fubini's theorem for elementary regions when the bounded domain D is an

- elementary region of type I
  $\{(x,y) : a \leq x \leq b, \quad \varphi_1(x) \leq y \leq \varphi_2(x)\}$
- or of type II $\{(x,y) : c \leq y \leq d, \ \psi_1(y) \leq x \leq \psi_2(y)\}$

Basic Question : Which functions $f : D \to \mathbb{R}$, D bounded subset of $\mathbb{R}^2$, are integrable?

For example, can we say that every continuous bounded function $f : D \to \mathbb{R}$ is integrable?

The answer to this second question is unfortunately No!

Example : Consider $R = [0,1] \times [0,1]$ and $D = \{ (x,y) \in R : \text{both } x \,\&\, y \text{ are rational} \}$

Let $f : D \to \mathbb{R}$ be the constant function

$$f(x,y) = 1 \quad \forall\ (x,y) \in D.$$

As a function on D,

f is clearly continuous

but its extension

$f^*: R \longrightarrow \mathbb{R}$

is the Dirichlet function

$$f^*(x,y) = \begin{cases} 1 & \text{if } (x,y) \in R \cap \mathbb{Q}^2, \\ 0 & \text{otherwise} \end{cases}$$

and this is not integrable on R! Hence, by defn., f is not integrable over D.

The problem here is that f^* is discontinuous on the boundary of D (denoted ∂D) and the boundary of D is TOO BIG! Indeed, $\partial D = R$.

A Comforting Fact: If the boundary of $D \subseteq \mathbb{R}^2$ is not too big and $f: D \to \mathbb{R}$ is continuous, then f is integrable over D.

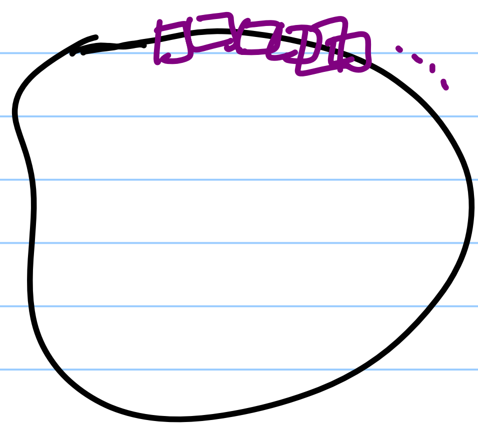
To make the idea of "too big" and "not too big" more precise, we introduce the following

Definition: Let E be a bounded subset of $\mathbb{R}^2$. We say that E is of (two-dimensional) content zero if the following condition holds:

For every $\varepsilon > 0$, E can be covered by finitely many rectangles $R_1, \ldots, R_k$ such that

$$E \subseteq \bigcup_{i=1}^{k} R_i \quad \text{and} \quad \sum_{i=1}^{k} \text{area}(R_i) < \varepsilon$$

Example:



① Any finite subset of $\mathbb{R}^2$ is of content zero.

② The graph of a Riemann-integrable function is of content zero, i.e., if

$$\varphi : [a,b] \longrightarrow \mathbb{R}$$

is R-integrable, then the set

$$E = \{ (x, \varphi(x)) : x \in [a,b] \}$$

is of content zero.

[To see this, note that by Riemann condition, for every $\varepsilon > 0$, $\exists$ a partition

$$P = \{a = x_0, x_1, \ldots, x_n = b\} \text{ of } [a,b]$$

such that

$$U(P,\varphi) - L(P,\varphi) < \varepsilon$$

This implies

$$E \subseteq \bigcup_{i=1}^{n} \underbrace{[x_{i-1}, x_i] \times [m_i(\varphi), M_i(\varphi)]}_{R_i}$$

and

$$\sum_{i=1}^{n} \text{area}(R_i) = U(P,\varphi) - L(P,\varphi) < \varepsilon \text{ .]}$$

Ex ③ : The set $\{(\frac{1}{n}, \frac{1}{k}) : n, k \in \mathbb{N}\}$ is of content zero.

Theorem: Let D be a bounded subset of $\mathbb{R}^2$ and $f: D \to \mathbb{R}$ be a bounded function.

If

1. the set of discontinuities of f is of content zero, and
2. ∂D, i.e., the boundary of D, is of content zero

then

f is integrable on D.

In particular, if f is continuous on D and ∂D is of content zero, then f is integrable on D.

[Proof skipped!]

Fubini's Theorem revisited:

IF D is an elementary region given by

$$D = \{ (x,y) \in \mathbb{R}^2 : a \leq x \leq b \,,\ \varphi_1(x) \leq y \leq \varphi_2(x) \}$$

where $a, b \in \mathbb{R}$ with $a < b$ and $\varphi_1, \varphi_2 : [a,b] \longrightarrow \mathbb{R}$ are bounded functions with at most finitely many discontinuities, and if $f : D \longrightarrow \mathbb{R}$ is a bounded function such that the set of all discontinuities of f on D is of content zero,

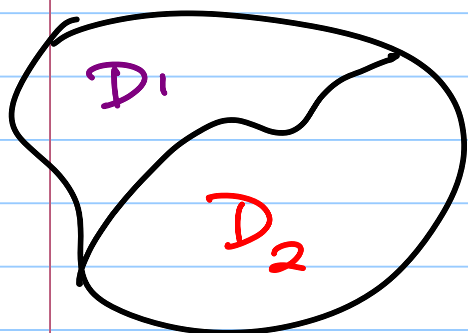
THEN

f is integrable over D and

$$\iint_D f(x,y)\, d(x,y) = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy \right) dx.$$

– A similar result holds for elementary regions of type II.

Domain additivity of double integrals



Let D be a bounded subset of $\mathbb{R}^2$ and let D_1, D_2 be subsets of D s.t.

- $D = D_1 \cup D_2$
- $D_1 \cap D_2$ is of content zero

If $f : D \to \mathbb{R}$ is a bounded function such that f is integrable over D_1 & over D_2, then f is integrable over D and

$$\iint_D f = \iint_{D_1} f + \iint_{D_2} f .$$

Remark: Fubini + domain additivity is a very useful and powerful tool for computing double integrals.

Another very useful tool is

Change of Variables

Recall that for Riemann integrals, we have

$$\int_{\varphi(c)}^{\varphi(d)} f(x)\,\underbrace{dx} = \int_{c}^{d} f(\varphi(t))\,\underbrace{\varphi'(t)\,dt}$$

provided (e.g.) $f : [a,b] \to \mathbb{R}$ is continuous and $\varphi : [c,d] \to [a,b]$ is continuously differentiable. and $\varphi'(t) \neq 0 \ \forall t$

Remark: The hypothesis on φ ensures that φ is 1-1 and in fact, φ' doesn't change sign throughout $[c,d]$. Consequently φ is either strictly increasing ($\leftrightarrow \varphi'(t) > 0 \ \forall t \in [c,d]$) or strictly decreasing ($\leftrightarrow \varphi'(t) < 0 \ \forall t \in [c,d]$.)

Hence

$$\varphi([c,d]) = [\varphi(c), \varphi(d)] \text{ or } [\varphi(d), \varphi(c)].$$

Thus if we assume that

$$\varphi([c,d]) = [a,b]$$

then we can replace the formula

$$\int_{\varphi(c)}^{\varphi(d)} f(x)\,dx = \int_c^d f(\varphi(t))\,\varphi'(t)\,dt$$

by the following:

$$\int_a^b f(x)\,dx = \int_c^d f(\varphi(t))\,|\varphi'(t)|\,dt.$$

To obtain a two-variable analogue of this one needs the following important notion:

Let Ω be an open subset of $\mathbb{R}^2$ and

$$\begin{aligned} \Phi = (\varphi_1, \varphi_2) : \Omega &\longrightarrow \mathbb{R}^2 \\ (u,v) &\longmapsto (\varphi_1(u,v), \varphi_2(u,v)) \end{aligned}$$

be a vector-valued function whose component functions φ_1, φ_2 have continuous partial derivatives. Then the <u>Jacobian</u> of Φ is

$$J(\Phi) = \frac{\partial(\varphi_1, \varphi_2)}{\partial(u,v)} := \det \begin{pmatrix} \frac{\partial \varphi_1}{\partial u} & \frac{\partial \varphi_1}{\partial v} \\ \frac{\partial \varphi_2}{\partial u} & \frac{\partial \varphi_2}{\partial v} \end{pmatrix}.$$

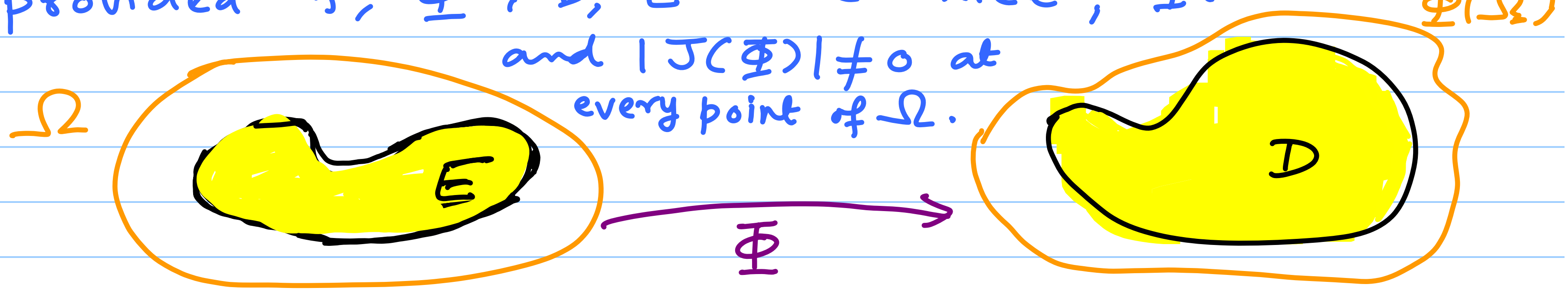
Theorem (Change of Variables Formula)

$$\iint_D f(x,y)\, d(x,y) = \iint_E (f \circ \Phi)(u,v)\, |J(\Phi)|\, d(u,v).$$

i.e.,

$$\iint_D f(x,y)\, d(x,y) = \iint_E f(\varphi_1(u,v), \varphi_2(u,v)) \left| \frac{\partial(\varphi_1,\varphi_2)}{\partial(u,v)} \right| d(u,v)$$

provided f, Φ, D, E are "nice", $\Phi(E)=D$, and $|J(\Phi)| \neq 0$ at every point of Ω.



Before we explain what is "nice", let us look at some

Examples: Let

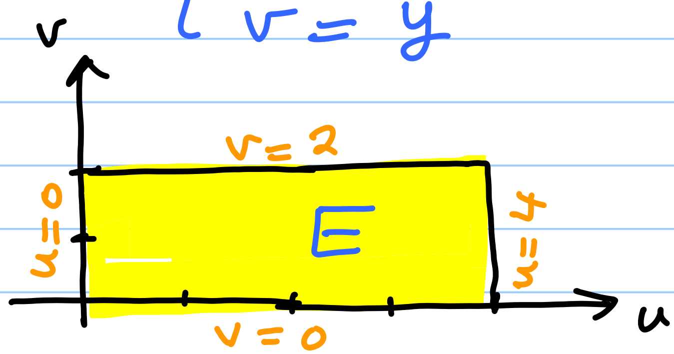
$$D = \{ (x,y) \in \mathbb{R}^2 : 0 \le y \le 2, \ \frac{y}{2} \le x \le \frac{y+4}{2} \}$$

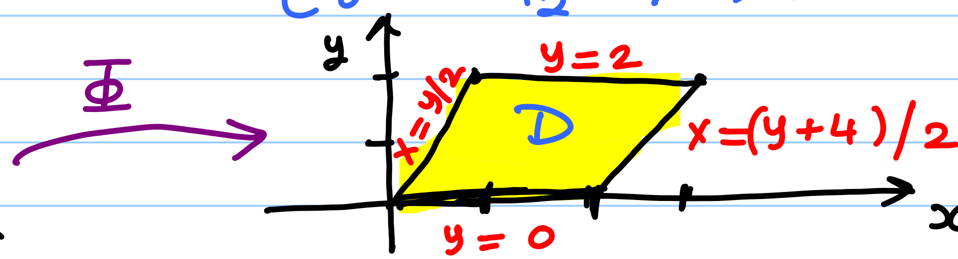
and

$$f(x,y) = y^3 (2x-y) \, e^{(2x-y)^2} \quad \text{for } (x,y) \in D$$

Consider the change of variables given by

$$\begin{cases} u = 2x - y \\ v = y \end{cases} \longleftrightarrow \begin{cases} x = \varphi_1(u,v) = \dfrac{u+v}{2} \\ y = \varphi_2(u,v) = v \end{cases}$$





Note that

$$J(\Phi) = \frac{\partial(\varphi_1, \varphi_2)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2} \neq 0$$

$\forall u, v$

and if

$$E = \{ (u,v) \in \mathbb{R}^2 : \quad 0 \leq u \leq 4 \;\; \& \;\; 0 \leq v \leq 2 \}$$

then $\Phi(E) = D$ and so by the Change of Variables formula,

$$\iint_D f(x,y)\, d(x,y) = \iint_E v^3 u \, e^{u^2} \cdot \frac{1}{2} \, d(u,v)$$

and hence by Fubini's theorem,

$$\iint_D f = \int_0^1 \left(\int_0^4 v^3 u\, e^{u^2} \cdot \frac{1}{2}\, du \right) dv$$

$$= \frac{1}{2} \int_0^1 v^3 \left(\frac{e^{16} - 1}{2} \right) dv$$

$$= e^{16} - 1.$$

2 Let $D = \left\{ (x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$.

Consider $f : D \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = y^2.$$

Consider a change of variables from rectangular to "polar coordinates", i.e.,

$$x = a r \cos\theta = \varphi_1(r,\theta)$$
$$y = b r \sin\theta = \varphi_2(r,\theta)$$

The Jacobian of $\Phi = (\varphi_1, \varphi_2)$ is given by

$$J(\Phi) = \frac{\partial(\varphi_1,\varphi_2)}{\partial(r,\theta)} = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr \neq 0$$

$\forall\ (r,\theta)$ in $(0,1] \times [0,2\pi]$

Thus the Jacobian is nonzero "almost everywhere" and hence by (a suitable version of the) Change of Variables formula,

$$\iint_D f = \iint_E (br\sin\theta)^2\, r\,ab\; d(r,\theta) = \int_0^1 \left(\int_0^{2\pi} \ldots d\theta \right) dr = ab^2\pi/4.$$

Precise Statement of the Change of Variables Formula

Theorem 1: Let D be a closed and bounded subset of $\mathbb{R}^2$ such that ∂D is of content zero, and let $f : D \to \mathbb{R}$ be a continuous function. Suppose

$$\begin{aligned} \Phi : \Omega &\longrightarrow \mathbb{R}^2 \\ (u,v) &\longmapsto (\varphi_1(u,v), \varphi_2(u,v)) \end{aligned}$$

is a one-one transformation from an open subset Ω of $\mathbb{R}^2$ such that $D \subseteq \Phi(\Omega)$ and such that φ_1, φ_2 have continuous partial derivatives in Ω and $J(\Phi)(u,v) \neq 0$ for all $(u,v) \in \Omega$. Then

$$\boxed{\iint_D f(x,y)\, d(x,y) = \iint_E (f \circ \Phi)(u,v)\, |J(\Phi)(u,v)|\, d(u,v),}$$

where E is a closed & bounded subset of Ω such that $\Phi(E) = D$.

Remark: The hypothesis that f is continuous on D can be replaced by the following weaker hypothesis:

- $f: D \to \mathbb{R}$ is bounded and its set of discontinuities is of content zero; moreover, the same is true for $f \circ \Phi : E \to \mathbb{R}$.

Theorem 2 (Changing to polar coordinates):
Let D be a closed and bounded subset of $\mathbb{R}^2$ such that ∂D is of content zero, and let $f: D \to \mathbb{R}$ be a continuous function. Suppose

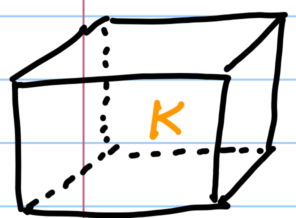
$$E := \{(r,\theta) \in \mathbb{R}^2 : r \geq 0, -\pi \leq \theta \leq \pi, (r\cos\theta, r\sin\theta) \in D\}$$

and suppose ∂E is also of content zero. Then

$$\boxed{\iint_D f(x,y)\, d(x,y) = \iint_E f(r\cos\theta, r\sin\theta)\, r\, d(r,\theta).}$$

Triple integrals

The theory of double integrals extends readily to that of triple integrals of functions of three variables. One proceeds along the same lines; here are the key steps



- first define triple integrals of bounded real-valued functions on cuboids, i.e., subsets of $\mathbb{R}^3$ of the form
$$K = [a, b] \times [c, d] \times [p, q]$$
where $a, b, c, d, p, q \in \mathbb{R}$ with $a < b$, $c < d$, $p < q$
[for this, one would consider upper triple sums, lower triple sums w.r.t. a partition of K etc.]

- next one observes that algebraic & order properties hold, and also that continuous functions on cuboids are triple integrable (whereas there do exist functions, such as the trivariate Dirichlet function that are not integrable)

- Fubini's theorem holds for integrable functions on cuboids, e.g., if $f : K \to \mathbb{R}$ is integrable, then

$$\iiint_K f(x,y,z)\, d(x,y,z) = \int_a^b \left(\iint_{[c,d]\times[p,q]} f(x,y,z)\, d(y,z) \right) dx$$

and

$$\text{''} = \iint_{[a,b]\times[c,d]} \left(\int_p^q f(x,y,z)\, dz \right) d(x,y)$$

provided the inner integrals exist. Consequently

$$\iiint_K f(x,y,z)\,d(x,y,z) = \int_a^b \left[\int_c^d \left(\int_p^q f(x,y,z)\,dz \right) dy \right] dx$$

[There are other versions with the role of x, y, z interchanged.]

— next, one extends the definition to bounded functions $f : D \to \mathbb{R}$, where D is a bounded subset of $\mathbb{R}^3$ by choosing a cuboid K containing D and extending f to K by

$$f^*(x,y,z) = \begin{cases} f(x,y,z) & \text{if } (x,y,z) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is said to be <u>integrable</u> over D if f^* is integrable on K and in this case,

$$\boxed{\iiint_D f \;:=\; \iiint_K f^* .}$$

- again algebraic & order properties extend easily
- Continuous functions $f : D \to \mathbb{R}$ will be integrable provided the boundary of its domain, i.e. ∂D, is of three-dimensional content zero.
- Fubini's theorem for "elementary regions" takes the following form:

Cavalieri's principle: Let $D \subseteq \mathbb{R}^3$ be bounded and $f : D \to \mathbb{R}$ be integrable over D.

① If $D = \{ (x, y, z) \in \mathbb{R}^3 : a \leq x \leq b \ \& \ (y, z) \in D_x \}$ where for each $x \in [a, b]$, the planar slice D_x is a subset of $\mathbb{R}^2$ such that ∂D_x is of (two-dimensional) content zero and the double integral $\iint_{D_x} f(x,y,z)\,d(y,z)$ exists. Then

$$\boxed{\iiint_D f(x,y,z)\,d(x,y,z) = \int_a^b \left(\iint_{D_x} f(x,y,z)\,d(y,z) \right) dx .}$$

② If D is a solid between two surfaces, i.e.,

$$D = \{ (x,y,z) \in \mathbb{R}^3 : (x,y) \in D_0 \ \& \ \varphi_1(x,y) \le z \le \varphi_2(x,y) \}$$

where $D_0 \subseteq \mathbb{R}^2$ with ∂D_0 of content zero and where $\varphi_1, \varphi_2 : D_0 \to \mathbb{R}$ are integrable functions s.t. $\varphi_1 \le \varphi_2$

Then

$$\boxed{\iiint_D f = \iint_{D_0} \left(\int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\,dz \right) d(x,y) .}$$

provided the inner integral exists.

Pictures copied from [GL-2] to illustrate the two parts of Cavalieri's Principle for Triple Integrals

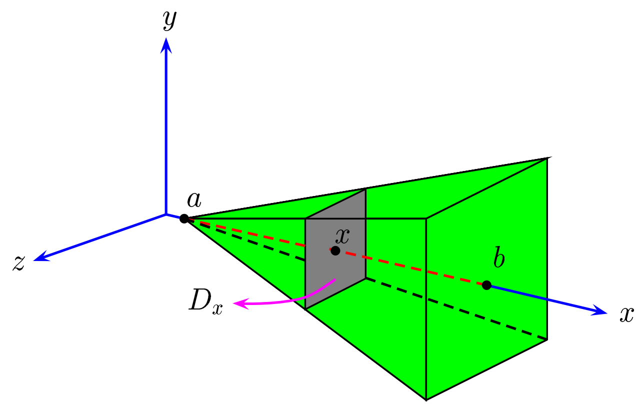


Figure 1: Illustration of Cavalieri's Principle (i): the slice D_x of a solid D.

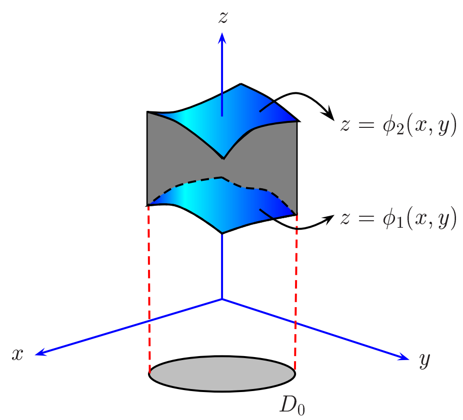


Figure 2: Illustration of Cavalieri's Principle (ii): a solid between two surfaces defined over D_0.

— the change of variables formula holds

[the statement of Theorem 1 is completely analogous and left to you to formulate; the key equation being

$$\iiint_D f(x,y,z)\, d(x,y,z) = \iiint_E (f\circ\Phi)(u,v,w)\, |J(\Phi)(u,v,w)|\, d(u,v,w)$$

where $J(\Phi)$ is the (3-dim'l) Jacobian of the transformation $\Phi = (\varphi_1, \varphi_2, \varphi_3) : \Omega \to \mathbb{R}^3$ given by

$$J(\Phi) = \det\begin{pmatrix} \frac{\partial\varphi_1}{\partial u} & \frac{\partial\varphi_1}{\partial v} & \frac{\partial\varphi_1}{\partial w} \\ \frac{\partial\varphi_2}{\partial u} & \frac{\partial\varphi_2}{\partial v} & \frac{\partial\varphi_2}{\partial w} \\ \frac{\partial\varphi_3}{\partial u} & \frac{\partial\varphi_3}{\partial v} & \frac{\partial\varphi_3}{\partial w} \end{pmatrix} = \frac{\partial(\varphi_1, \varphi_2\, \varphi_3)}{\partial(u,v,w)}.$$

Theorem 2 has the following analogue with two parts.

<u>Theorem 2'</u> : Let $D \subseteq \mathbb{R}^3$ be closed & bounded s.t. ∂D is of three-dimensional content zero and let $f : D \to \mathbb{R}$ be continuous.

(i) [Changing to cylindrical coordinates] If

$$E = \left\{ (r, \theta, z) \in \mathbb{R}^3 : \begin{array}{l} r \geq 0, \ -\pi \leq \theta \leq \pi, \text{ and} \\ (r\cos\theta, \ r\sin\theta, \ z) \in D \end{array} \right\}$$

and if ∂E is of three-dimensional content zero, then

$$\iiint_D f(x, y, z)\, d(x, y, z) = \iiint_E f(r\cos\theta, r\sin\theta, z)\, r\, d(r, \theta, z).$$

(ii) [Changing to spherical coordinates] If

$$E = \{ (\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho \geq 0,\ 0 \leq \varphi \leq \pi,\ -\pi \leq \theta \leq \pi, \text{ and } (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho\cos\varphi) \in D \}$$

and if ∂E is of three-dimensional content zero, then

$$\iiint_D f(x,y,z)\, d(x,y,z) = \iiint_E f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) \times \rho^2 \sin\varphi\, d(\rho,\varphi,\theta).$$

For further details, one may refer to [GL-2] and the references therein, if interested.

Tut sheet No. 8 has some examples of triple integrals that you are encouraged to attempt.