Surface Integrals and Stokes' Theorem

Note Title 11/4/2011

Recall that we have discussed the notion of

- parametrized surface

  [smooth map $\Phi : E \to \mathbb{R}^3$, $E \subseteq \mathbb{R}^2$ has area

  $(u,v) \longmapsto \Phi(u,v) = (x(u,v), y(u,v), z(u,v))$]

- normal vector and unit normal to Φ at $\Phi(u_0,v_0)$

  $(\Phi_u \times \Phi_v)(u_0,v_0)$ and $\dfrac{\Phi_u \times \Phi_v(u_0,v_0)}{|\Phi_u \times \Phi_v(u_0,v_0)|}$

  – provided $\Phi_u \times \Phi_v(u_0,v_0) \neq 0$.

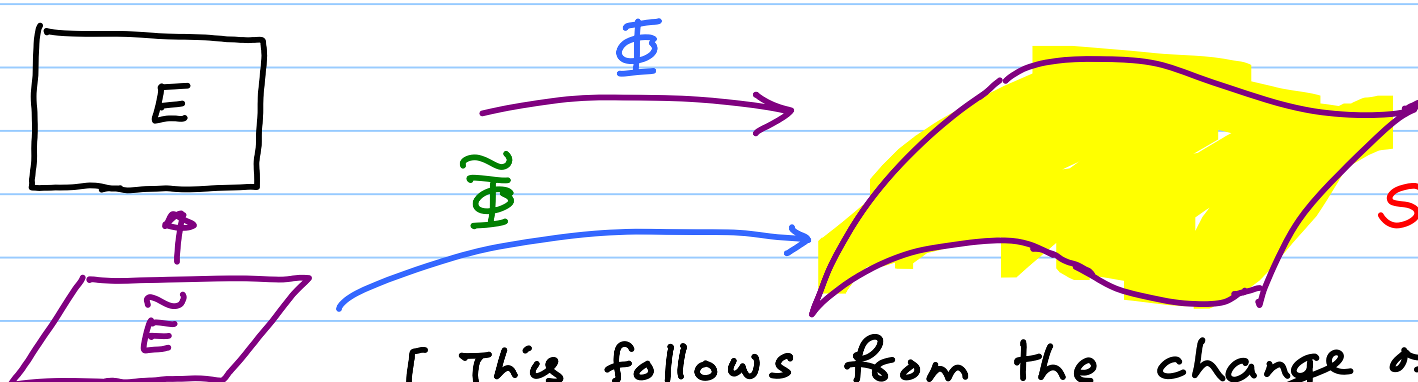
- regular parametrization

  – unit normal exists at every point

- surface area

  – $\iint_E |\Phi_u \times \Phi_v| \, d(u,v) = \iint_E dS$ or $\iint_E d\sigma$

We noted that the surface area is invariant under reparametrizations. More precisely, if $\Phi : E \to \mathbb{R}^3$ is a parametrized surface and $h : \tilde{E} \to E$ is one-one, onto and smooth function such that the Jacobian $J(h) \neq 0$, then for the parmetrized surface $\tilde{\Phi} : \tilde{E} \to \mathbb{R}^3$ defined by $\tilde{\Phi} = \Phi \circ h$ we have $\text{Area}(\Phi) = \text{Area}(\tilde{\Phi})$



[This follows from the change of variables formula]

With this in view, if we let

$$S = \Phi(E)$$

to be the (geometric) surface in $\mathbb{R}^3$ determined by the parametrization $\Phi : E \to \mathbb{R}^3$ then we define

$$\text{Surface area of } S = \text{Surface area of } \Phi = \iint_E |\Phi_u \times \Phi_v| \, d(u,v)$$

or simply

$$\text{surface area of } S = \iint_S dS = \iint_E |\Phi_u \times \Phi_v| \, d(u,v) .$$

We have also defined surface integrals of scalar fields and in a similar way one sees that these are independent of reparametrization and so we may write for a scalar field on $D \supseteq \Phi(E)$,

$$\iint_S f\, dS := \iint_E f(\Phi(u,v))\, |\Phi_u \times \Phi_v|\, d(u,v).$$

<u>Example</u> Consider the upper hemisphere H given by

$$x^2+y^2+z^2 = a^2, \quad z \geq 0$$

Find the surface area of H and $\iint_H z\, dS$.

We have seen that for the parametrization Φ in terms of spherical coordinates

$$|\Phi_u \times \Phi_v| = a^2 |\sin u|$$

hemisphere $\longleftrightarrow$ $(u,v) \in [0, \pi/2] \times [0, 2\pi]$

and so

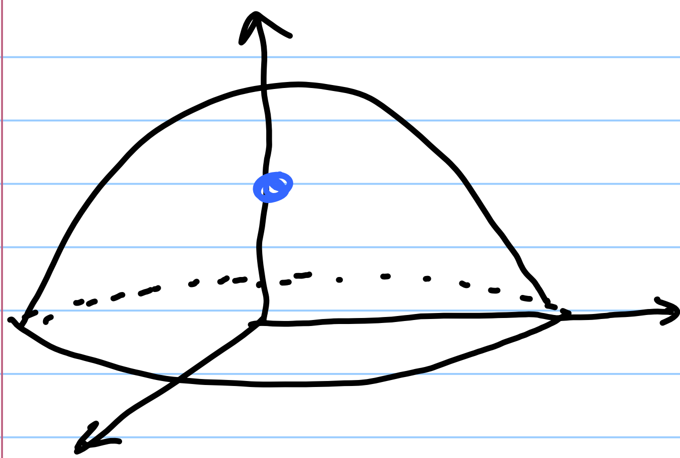
$$\text{Area}(H) = \iint_H dS = \int_0^{\pi/2}\int_0^{2\pi} a^2 \sin u\, dv\, du = 2\pi a^2$$

Next

$$\iint_H z\,dS = \int_0^{\pi/2}\int_0^{2\pi} (a\cos u)(a^2\sin u)\,dv\,du$$

$$= \pi a^3 \int_0^{\pi/2} \sin 2u\,du \quad = \pi a^3.$$

Observe that this example shows that the centroid of the hemisphere H is at



$$(\bar{x}, \bar{y}, \bar{z}) := \left(\frac{\iint x\,dS}{\iint dS}, \frac{\iint y\,dS}{\iint dS}, \frac{\iint z\,dS}{\iint dS} \right)$$

$$= (0, 0, a/2)$$

since $\bar{x} = \bar{y} = 0$, by symmetry.

Surface integrals of vector fields, unlike those of scalar fields, can however depend on the parametrization. Recall the definition:

$$\iint_{\Phi} \vec{F} \cdot d\vec{S} = \iint_{E} \vec{F}(\Phi(u,v)) \cdot (\Phi_u \times \Phi_v)(u,v)\, d(u,v)$$

or equivalently when Φ is regular,

$$\iint_{\Phi} \vec{F} \cdot d\vec{S} = \iint_{\Phi} \vec{F} \cdot \hat{n}\, dS$$

where

$$d\vec{S} = (\Phi_u \times \Phi_v)\, d(u,v) \qquad \& \qquad \hat{n} = \frac{\Phi_u \times \Phi_v}{|\Phi_u \times \Phi_v|}$$

$$dS = |\Phi_u \times \Phi_v|\, d(u,v)$$

Now at each point of the (geometric) surface

$$S = \Phi(E)$$

we have two choices for a unit normal ("outward" or "inward"). For example, for the sphere

$$x^2 + y^2 + z^2 = a^2$$

we may take either

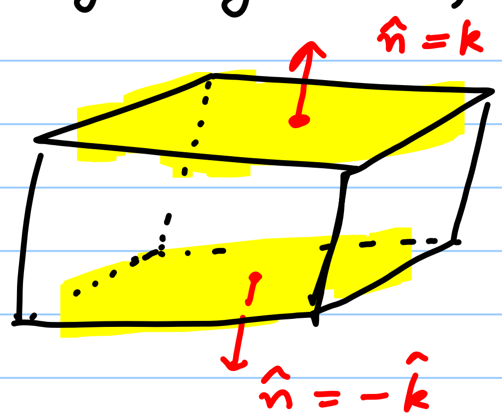
$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} \quad \text{or} \quad \hat{n} = \frac{-(x\hat{i} + y\hat{j} + z\hat{k})}{a}$$

If we take either of the alternatives uniformly then we have two possibilities for the surface integral of a vector field on the sphere, which differ in sign.

Defn: A surface S in $\mathbb{R}^3$ is said to be <u>orientable</u> if there exists a unit normal vector $\hat{n}(P)$ at each point $P \in S$ which is a continuous function of P.

By an <u>oriented</u> surface we mean an orientable surface with a fixed choice of orientation.

Most of the surfaces we have come across (e.g. cylinder, sphere) are orientable. Sometimes the surface may comprise of finitely many orientable surfaces with "thin" overlaps (of two-dim'l content zero) such as a cube or a cuboid. This may still be regarded as orientable



Fact : There do exist nonorientable surfaces

A classic example is the Möbius strip.
Explicitly, this is given by

$$\Phi(u,v) = \left(1+\frac{1}{2}v\cos\frac{u}{2}\right)\cos u\,\hat{i} + \left(1+\frac{1}{2}v\cos\frac{u}{2}\right)\sin u\,\hat{j} + \frac{1}{2}v\sin\frac{u}{2}\,\hat{k}$$

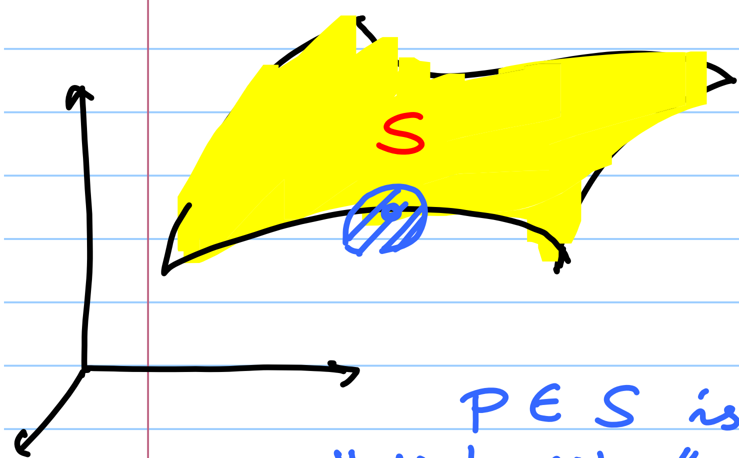
for $(u,v) \in [0,2\pi] \times [-1,1]$.

[Use a web search for nice pictures.]

Boundary of a surface

Suppose S is a (geometric) surface in $\mathbb{R}^3$. Assume that S is closed.

Now typically, as a subset of $\mathbb{R}^3$, no point of S is an interior point and thus it would seem that $\partial S = S$ in accordance with the earlier definition. There is, however, an intrinsic notion of boundary of S which may be 'defined' as follows.



$P \in S$ is an interior point of S if there is a "disk-like" neighbourhood of P contained in S; otherwise P is a boundary point of S.

Notation: We will still use ∂S to denote the boundary of S in the above sense.

Examples: S = sphere $\rightsquigarrow$ $\partial S = \emptyset$

S = cube $\rightsquigarrow$ $\partial S = \emptyset$

S = cylindrical can $\{x^2+y^2=a^2,\ 0\le z\le h\} \cup \{x^2+y^2\le a^2,\ z=0\}$ $\longrightarrow$ boundary = top circle $x^2+y^2=a^2$, $z=h$

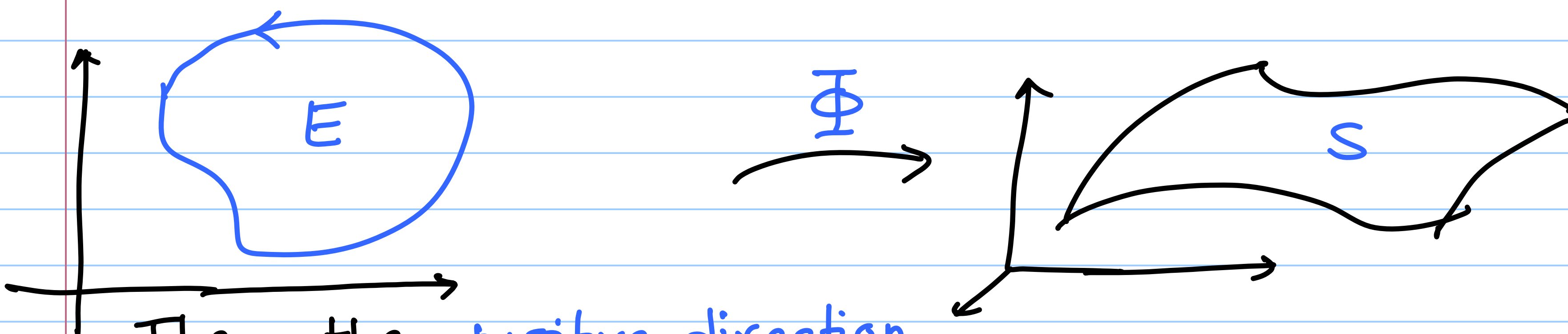
Remark: Surfaces S with $\partial S=\emptyset$ are sometimes called closed surfaces. A better nomenclature is surface without boundary.

Fact: Oriented surfaces induce an orientation of the boundary, referred to as the induced orientation.

Suppose S is a surface with a regular parametrization

$$\Phi : E \longrightarrow \mathbb{R}^3$$

where E is a nice subset of $\mathbb{R}^2$



Then the positive direction of ∂E corresponds to the induced orientation on ∂S by the choice of unit normal vector given by

$$\hat{n} = \frac{\Phi_u \times \Phi_v}{|\Phi_u \times \Phi_v|}.$$

Stokes' Theorem

Let S be a piecewise C^2 bounded oriented surface in $\mathbb{R}^3$ whose boundary ∂S has the induced orientation. If $\vec{F}$ is a smooth vector field defined on an open subset containing S and ∂S, then

$$\boxed{\oint_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S (\operatorname{curl} \vec{F}) \cdot \hat{n} \, dS.}$$

Note: Stokes' Theorem may be thought of as the "curved" version of Green's theorem.

Sketch of Proof :

Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ so that

$$\oint_{\partial S} \vec{F} \cdot d\vec{s} = \int_{\partial S} P\,dx + Q\,dy + R\,dz$$

By Chain Rule,

$$\int_{\partial S} P\,dx = \int_{\partial E} P(\Phi(u,v))\left(\frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv\right)$$

So if we write $h(u,v) = P(\Phi(u,v))$ then the RHS above is

$$\int_{\partial E} (h\,x_u)\,du + (h\,x_v)\,dv$$

We now use Green's theorem to obtain

$$\int_{\partial S} P\,dx = \iint_E \left(\frac{\partial}{\partial u}(h x_v) - \frac{\partial}{\partial v}(h x_u) \right) d(u,v)$$

$$= \iint_E \left[h_u x_v + \cancel{h x_{vu}} - h_v x_u - \cancel{h x_{uv}} \right] d(u,v)$$

$$= \iint_E (h_u x_v - h_v x_u)\, d(u,v)$$

Now by Chain Rule applied to $h(u,v) = P(\Phi(u,v))$, the partial derivatives h_u, h_v can be expanded to write the above integral as

$$\iint_E \left\{ (\cancel{\underbrace{P_x x_u}}+ P_y y_u + P_z z_u) x_v - (\cancel{\underbrace{P_x x_v}} + P_y y_v + P_z z_v) x_u \right\} d(u,v)$$

Thus,

$$\int_{\partial S} P\,dx = \iint_E -P_y \frac{\partial(x,y)}{\partial(u,v)} + P_z \frac{\partial(z,x)}{\partial(u,v)}\, d(u,v)$$

In a similar manner, we see that

$$\int_{\partial S} Q\,dy = \iint_E -Q_z \frac{\partial(y,z)}{\partial(u,v)} + Q_x \frac{\partial(x,y)}{\partial(u,v)}\, d(u,v)$$

and

$$\int_{\partial S} R\,dz = \iint_E -R_x \frac{\partial(z,x)}{\partial(u,v)} + R_y \frac{\partial(y,z)}{\partial(u,v)}\, d(u,v)$$

Hence

$$\int_{\partial S} \vec{F}\cdot d\vec{s} = \iint_E \left[(Q_x - P_y)\frac{\partial(x,y)}{\partial(u,v)} + (P_z - R_x)\frac{\partial(z,x)}{\partial(u,v)} + (R_y - Q_z)\frac{\partial(y,z)}{\partial(u,v)}\right] d(u,v)$$

To complete the proof, it suffices to observe that

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{i}$$

and as noted in the previous lecture,

$$(\Phi_u \times \Phi_v) = \frac{\partial(y,z)}{\partial(u,v)}\hat{i} + \frac{\partial(z,x)}{\partial(u,v)}\hat{j} + \frac{\partial(x,y)}{\partial(u,v)}\hat{k}$$

It follows that

$$\oint_{\partial S} \vec{F}\cdot d\vec{s} = \int_{\partial S} P\,dx + Q\,dy + R\,dz = \iint_S (\operatorname{curl} \vec{F})\cdot d\vec{S},$$

as desired.

Remark: As an application of Stokes' Theorem one can show that if $\vec{F}$ is a smooth vector field on a simply-connected domain in $\mathbb{R}^3$ such that curl $\vec{F} = \vec{0}$, then $\vec{F}$ is conservative. More applications & examples will be discussed later.