Lecture 1 06/05/09 MA 205 Complex Analysis:

Notes by BANOTHU. RAGHU

6 - May - 3 July 3 1 0 4

1. R.V. Churchill & J.W. Brown
   Complex Variables and applications (7th edition), 2003.

2. J.M. Howie, Complex Analysis, 2004.

3. T.W. Gamelin, Complex Analysis, 2001

4. J.B. Conway, " "

5. L.V. Ahlfors, "

Background expected:

Basics of calculus/Real Analysis.

functions

limits

continuity

differentiability

integration.

Ref: ACICARA (A course in calculus & Real analysis. Ghorpade & Limaye).

Complex Numbers:

Def:

A complex ~~re~~ number is a pair (a,b) of real numbers a, b.

Def: $i = (0,1)$.

Addition and multiplication of complex numbers is defined as follows,

$$(a,b) + (c,d) = (a+c, b+d)$$

$$(a,b) \cdot (c,d) = (ac-bd, ad+bc)$$

We identify the elements of the form $(a,0)$ with the real number a

[Note:

$$(a,0) + (c,0) = (a+c, 0)$$

$$(a,0) \cdot (c,0) = (ac, 0) \quad \forall a, c \in \mathbb{R}]$$

With this in view, the pair (a,b) can be written as $a+bi$.

Note: $\Rightarrow (a,0) + (b,0)(0,1) = (a,0) + (b.0-0, b.1+0.0)$

$$= (a,0) + (0,b).$$

$$= (a,b).$$

$$\Rightarrow i.i = (0,1)(0,1) = (-1,0) = -1.\}$$

We can then write $(*)$ as,

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$$

We let,

$\mathbb{C}$ = the set of all complex numbers.

$= \{a+bi : a, b \in \mathbb{R}\}.$

Def:

If $z = a+bi$ is a complex number, then a and b are uniquely determined by z and are called the real part and the imaginary part of z respectively.

Notation:

$$Re(z) = a \quad ; \quad Im(z) = b.$$

Def:

The modulus or the absolute value of a complex number $z = a+bi$ is $|z| = \sqrt{a^2+b^2}$.

The conjugate of z is $\bar{z} = a-bi$

Note:

For $z, \omega \in \mathbb{C}$, we have:

→ $z = \omega \iff Re(z) = Re(\omega)$ & $Im(z) = Im(\omega)$

→ $Re(z+\omega) = Re(z) + Re(\omega)$

$Im(z+\omega) = Im(z) + Im(\omega)$

→ $\overline{z+\omega} = \bar{z}+\bar{\omega}$; $\overline{zw} = \bar{z}\bar{w}$.

$\rightarrow$ $|z|^2 = z\bar{z}$ and hence $|zw| = |z||w|$; $|\bar{z}| = |z|$

Further,
$$|z+w| \leq |z| + |w|$$

$$\Big(|z+w|^2 = (z+w)(\bar{z}+\bar{w})$$
$$= |z|^2 + |w|^2 + z\bar{w} + \bar{z}w.$$
$$= |z|^2 + |w|^2 + 2\,Re(z\bar{w})$$
$$\leq |z|^2 + |w|^2 + 2|z\bar{w}|$$
$$\leq |z|^2 + |w|^2 + 2|z||w|$$
$$= (|z| + |w|)^2$$

Hence,
$$|z+w| \leq |z| + |w| \text{ \& equality holds}$$
$$\Leftrightarrow Im(z\bar{w}) = 0. \text{ \& } Re(z\bar{w}) \geq 0.$$

This is called triangle inequality.

Also we have,
$$\big||z| - |w|\big| \leq |z-w| \quad \forall\, z, w \in \mathbb{C}$$

$\Big\{$ Pf: $|z| = |z-w+w| \leq |z-w| + |w|$

so, $|z| - |w| \leq |z-w|$

$|||^{ly}$, $|w| - |z| \leq |z-w|$.

Hence,
$$\big||z| - |w|\big| \leq |z-w|.$$

Note that as a conseq. of triangle equality, we obtain,
$$|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|.$$

Cauchy - Schwarz Inequality:

If $z_1, \ldots z_n, w_1, \ldots w_n \in \mathbb{C}$, then,

$$\boxed{\left|\sum_{i=1}^{n} z_i w_i\right| \le \left(\sum_{i=1}^{n} |z_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |w_i|^2\right)^{1/2}}$$

Pf: Exercise.

Sequences in $\mathbb{C}$:-

Def: A sequence in $\mathbb{C}$ is a function from the set $\mathbb{N} = \{1, 2, 3, 4 \ldots\}$ to $\mathbb{C}$.

Usually, we write (z_n) or $\{z_n\}$ to denote a sequence. Here (z_n) corresponds to the function $\mathbb{N} \to \mathbb{C}$ which associates to $n \in \mathbb{N}$ the complex number z_n.

Eg:- $(\frac{1}{n})$, $((1+1/n)^n)$, ~~$(\frac{i^n}{n})$, (i/n)~~. (i^n), (i/n), (n^i)

Def:

A sequence (z_n) in $\mathbb{C}$ is said to be:

↳ bounded if $\exists$ $M \in \mathbb{R}$ s.t. $|z_n| \le M$ $\forall n \in \mathbb{N}$

↳ Convergent if $\exists$ $z \in \mathbb{C}$ s.t. for every $\varepsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ s.t. $|z_n - z| < \varepsilon$ $\forall n \ge n_0$.

Example:

Let $z_n = i/n$: for $n \in \mathbb{N}$,

we claim that (z_n) is convergent

In this case, the complex number z is unique (prove!) and is called the limit of (z_n) and denoted by $\lim_{n\to\infty} z_n$.

Eg: $z_n = i/n$.

$$\lim_{n \to \infty} z_n = 0.$$

Pf: Let $\varepsilon > 0$ be given,

To show: $\exists\, n_0 \in \mathbb{N}$ s.t. $\frac{1}{n} = \left|\frac{i}{n} - 0\right| < \varepsilon \;\forall\, n \geq n_0$.

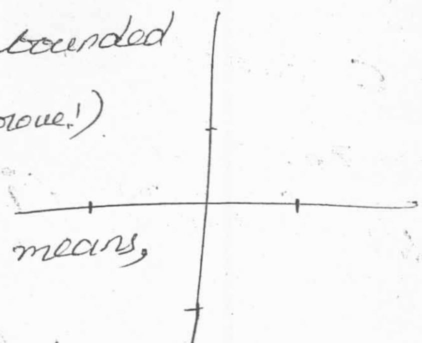
So if we let $n_0 = \left[\frac{1}{\varepsilon}\right] + 1$, then

$$n \geq n_0 \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon$$

$$\Rightarrow \left|\frac{i}{n} - 0\right| < \varepsilon$$

Eg: i^n

(2) The sequence (i^n) is bounded but not convergent. (prove!)

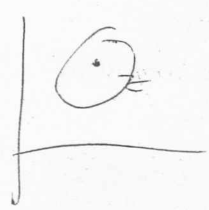


(z_n) is not convergent means,

$\forall\, z \in \mathbb{C}$ $\exists\, \varepsilon > 0$ ~~st~~ ~~such~~ such that for every $n_0 \in \mathbb{N}$, $\exists\, n \geq n_0$ such that $|z_n - z| > \varepsilon$.

However,

every convergent sequence is bounded.



Pf: Suppose $z_n \to z$

Take $\varepsilon = 1$, then $\exists\, n_0$, s.t.

$$|z_n - z| \leq 1 \quad \forall\, n \geq n_0$$

$$\Rightarrow |z_n| \leq |z| + 1 \quad \forall\, n \geq n_0.$$

So if $M = \max\{|z|+1, |z_1|, |z_2| \dots |z_{n_0-1}|\}$

then, $|z_n| \leq M \ \forall \ n \in \mathbb{N}$.

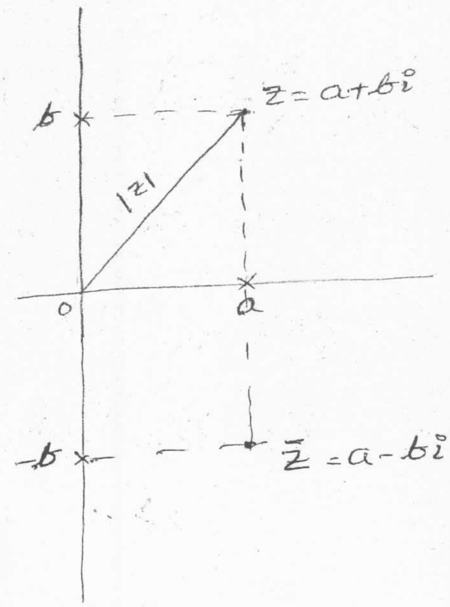
<u>Exercise</u>,

Show that $\mathbb{C}$ dosen't have "order Property".

<u>07/05/09</u>

Review:

$$\mathbb{C} = \{a+bi : a, b \in \mathbb{R}\}$$



For $z = a+bi \in \mathbb{C}$

$$|z| = \sqrt{a^2+b^2}$$

$$\bar{z} = a-bi$$

We showed

$\rightarrow$ $\overline{z+w} = \bar{z}+\bar{w}$; $\overline{zw} = \bar{z}\bar{w}$

$\rightarrow$ $|z+w| \leq |z|+|w|$

$\rightarrow$ $||z|-|w|| \leq |z-w|$

Cauchy-Schwarz Inequality:

For $z_1, \ldots z_n, w_1, \ldots w_n \in \mathbb{C}$,

$$\Rightarrow \left|\sum_{i=1}^{n} z_i w_i\right| \leq \left(\sum_{i=1}^{n} |z_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |w_i|^2\right)^{1/2} \longrightarrow (1)$$

Other forms of C-S Inequality are,

$$\sum_{i=1}^{n} |z_i w_i| \leq \left(\sum_{i=1}^{n} |z_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |w_i|^2\right)^{1/2} \longrightarrow (2)$$

$$\left|\sum_{i=1}^{n} z_i \bar{w}_i\right| \leq \quad " \quad " \longrightarrow (3)$$

Note: (2) $\Rightarrow$ (1) since $|\sum z_i w_i| \leq \sum |z_i w_i|$

(1) $\Rightarrow$ (2), follows by applying (1) to

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