MA 205 Complex Analysis.

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Lecture 10 Dated 21/05/09

Morera's Theorem (A converse of Cauchy's Theorem)

let Ω be a open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be continuous. If

$$\int_\gamma f(z)\, dz = 0$$ for every simple closed piecewise smooth path γ in Ω

then the function is analytic in Ω

<u>proof</u>

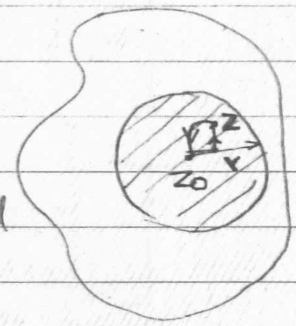
let $z_0 \in \Omega$

Since Ω is open $\exists\ r > 0$ such that $B(z_0, r) \subseteq \Omega$

We'll show: f is analytic on $D = B(z_0, r)$

Consider $F : D \to \mathbb{C}$ defined by

$$F(z) = \int_{z_0}^{z} f(w)\, dw.$$



where by $\int_{z_0}^{z}$ we mean intergral

$\int_\gamma f(w)\,dw$ where γ is any simple, piecewise smooth path from z_0 to z

Note that this is a well defined because if $\tilde{\gamma}$ is another such path then $\gamma - \tilde{\gamma}$ is a closed path in D and by hypothesis

$$\int_{\gamma-\tilde{\gamma}} f(w)\, dw = 0.$$

$$\Rightarrow \int_{\gamma} + \int_{-\tilde{\gamma}} = 0 \Rightarrow \int_{\gamma} = \int_{\tilde{\gamma}}$$

Thus $F : D \to \mathbb{C}$ is well defined.

we claim F is analytic on D & $F' = f$

proof :

let $z \in D$. Now

$$\lim_{h\to 0} \left| \frac{F(z+h) - F(z)}{h} - f(z) \right|$$

$$= \frac{1}{h}\left[\int_{z_0}^{z+h} f(w)\, dw - \int_{z_0}^{z} f(w)\, dw \right] - f(z).$$

$$= \frac{1}{h}\left[\int_{z}^{z+h} f(w)\, dw \right] - \frac{1}{h}\int_{z}^{z+h} f(z)\, dw$$

where the integral is along the line segment z to z+h

$$= \frac{1}{h}\int_{z}^{z+h} [f(w) - f(z)]\, dw$$

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Since f is continuous, given any $\varepsilon > 0$ $\exists\ \delta > 0$ such that

$$\omega \in \Omega \text{ and } |\omega - z| < \delta \Rightarrow |f(\omega) - f(z)| < \varepsilon$$

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$$0 < |h| < \delta \Rightarrow |f(\omega) - f(z)| < \varepsilon$$

$\forall\ \omega$ on the line segment from z to $z+h$.

$$\Rightarrow \left| \frac{1}{h} \int_{z}^{z+h} (f(\omega) - f(z))\, d\omega \right.$$

$$\leq \frac{1}{|h|} \int_{z}^{z+h} |f(\omega) - f(z)|\, |d\omega|$$

$$= \frac{1}{|h|} \int_{z}^{z+h} \varepsilon\, |d\omega|$$

$$= \frac{\varepsilon}{|h|} \cdot |h|$$

$$\Rightarrow \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \leq \varepsilon$$

So $\lim\limits_{h \to 0} \dfrac{F(z+h) - F(z)}{h}$ exist and is $f(z)$

$\therefore F(z)$ is differentiable at z and $F'(z) = f(z)$
Since z was an arbitrary point of D
F is analytic on D and $F' = f$ on D

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Since F is analytic $\Rightarrow$ F' is analytic
we conclude that f is analytic on D
Varying z_0 we find that f is analytic on Ω

SEQUENCES AND SERIES.

Recall: A sequence (z_n) in $\mathbb{C}$ is said to be convergent if there is $z \in \mathbb{C}$ with the property that for every $\varepsilon > 0$ $\exists\, n_0 \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon \qquad \forall\, n \geq n_0.$$

Infinite Series:

An infinite series is

$$\sum_{n=1}^{\infty} z_n$$

of complex numbers we define

$$S_n = \sum_{k=1}^{n} z_k$$

then (s_n) is a sequence in $\mathbb{C}$, called the sequence of partial sums of the series.

Def:

We say that the series $\sum z_n$ is convergent if the sequence (S_n) of its partial sums is convergent and in this case $\lim_{n\to\infty} S_n$ is called the sum of the series $\sum z_n$ and is also denoted by $\sum_{n=1}^{\infty} z_n$

Remark : A similar definition applies to series of the form $\sum_{n=0}^{\infty} z_n$

eg. Suppose $z \in \mathbb{C}$. Consider the series

$$\sum_{n=0}^{\infty} z^n$$

Here

$$S_n = 1 + z + z^2 + \dots + z_n$$

$$= \frac{z^{n+1} - 1}{z - 1}$$

If $|z| < 1$ then

$$|z^{n+1}| = |z|^{n+1} \to 0 \text{ as } n \to \infty \text{ and hence}$$

$S_n \to \frac{1}{z-1}$ provided $|z| < 1$

So the series $\sum_{n=0}^{\infty} z^n$ is convergent with sum $\frac{1}{z-1}$ when $|z| < 1$

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Property : If a series $\sum z_n$ is convergent then $z_n \to 0$ as $n \to \infty$

proof : $\therefore z_n = S_n - S_{n-1} \longrightarrow S - S = 0$.

where $S = \lim_{n\to\infty} S_n$.

eg. $\sum_{n=0}^{\infty} z^n$ is divergent (= not convergent)

if $|z| \geq 1$ because $|z| \geq 1 \Rightarrow |z^n| = |z^n| \geq 1$ $\forall n \Rightarrow z^n \not\to 0$

note the converse is not true i.e.

$$z_n \to 0 \not\Rightarrow \sum z_n \text{ is convergent}$$

eg. consider $z_n = \frac{1}{n}$

clearly $z_n \Rightarrow 0$ as $n \to \infty$, but

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

so the series $\sum \frac{1}{n}$ is divergent

to see it consider

$$S_{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \cdots$$

$$\geq 1 + \frac{1}{2} + \frac{1}{2} + \cdots$$

$$= 1 + \frac{n}{2} \to \infty \text{ as } n \to \infty$$

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Remark : On the other han

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent if } p>1$$

Also $\sum_{n=1}^{\infty} \left(\frac{-1}{n}\right)^{n+1}$ is convergent and converges to $\log_e 2$

Def :

A series $\sum z_n$ is said to be absolutely convergent if ~~f~~ $\sum |z_n|$ is convergent

Property : If $\sum z_n$ is absolutely convergent then it is convergent.

We'll proove this by using

Comparison test :

If $\sum z_n$ is a series real numbers such that

$$0 \cancel{u_n} \le z_n \le v_n \qquad \forall n$$

and both $\sum u_n$ and $\sum v_n$ are convergent then $\sum z_n$ is convergent and

$$0 \cancel{u} \le z < v$$

where

$\cancel{u = \sum u_n}$

$v = \sum v_n$

$z = \sum z_n.$

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proof: If ~~θ~~ S_n, V_n are seq. of partial sums of $\sum z_n$ & $\sum v_n$ resp. then

$$0 \le S_n \le V_n$$

moreover V_n is convergent $\Rightarrow$ V_n is bounded
$\Rightarrow$ S_n is bounded.

Also since $z_n \ge 0$ $\forall$ n, the sequence (S_n) is monotonically increasing. Hence S_n is convergent and so $\sum z_n$ is convergent

proof of previous property

Suppose $z_n \in \mathbb{C}$ are such that

$\sum |z_n|$ is convergent

To show: $\sum z_n$ is convergent

It suffice to show that

$\sum Re(z_n)$ and $\sum Im(z_n)$ are convergent.

Now

$$|Re(z_n)| \le |z_n|$$

and thus.

$$0 \le Re(z) + |z_n| \le 2|z_n|$$

Now $\sum |z_n|$ convergent $\Rightarrow$ $\sum 2|z_n|$ convergent

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and hence by comparison test

$$\sum Re(z_n) + |z_n| \quad \text{is convergent}$$

So $\sum Re(z_n) = \sum Re(z_n) + |z_n| - \sum |z_n|$ is convergent being the difference between two convergent series

Similarly

$\sum Im(z_n)$ is convergent

Hence $\sum z_n$ is convergent

Cauchy Criterion for Sequences.

A series $\sum z_n$ is convergent if and only if for every $\varepsilon > 0$ $\exists\ n_0 \in \mathbb{N}$ such that

$$\left| \sum_{k=m+1}^{n} z_k \right| < \varepsilon \quad \forall \quad n > m \geq n_0$$

proof follows from Cauchy Criterion for sequence since

$$S_n - S_m = \sum_{k=m+1}^{n} z_k \quad \forall \quad n > m \geq n_0 \in \mathbb{N}$$

Corollary : (property)

$\sum z_n$ is convergent $\Rightarrow$ the tail of $\sum z_n$ goes to 0

i.e $\sum_{k=m+1}^{\infty} z_k \to 0$ as $m \to \infty$

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Note if S is the sum of $\sum z_n$, then

$$\sum_{k=m+1}^{\infty} z_k = S - S_m$$

proof $\left|\sum_{k=m+1}^{\infty} z_k\right| = |S - S_m| \to 0$ as $m \to \infty$

since $S_m \to S$.