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lecture 11 Dated - 21/05/09.

Sequences and series of Functions :

let $\Omega \subseteq \mathbb{C}$. A sequence of functions on Ω is a map from the set $\mathbb{N}$ of positive integers into the set of all $\mathbb{C}$ valued functions on Ω.

Typically a seq. of functions is denoted by (f_n) where each term ~~is~~ f_n is a map from $\Omega \to \mathbb{C}$

eg.

f_n where $f_n(z)$ is $\frac{z}{n}$ for $z \in \Omega$

$f_n(z)$ is z^n

If f_n is a sequence of functions on Ω then for each $z \in \Omega$

$$(f_n(z))$$

is a sequence of complex numbers.

Def : A sequence (f_n) of function on $\Omega \subseteq \mathbb{C}$ is said to be <u>pointwise convergent</u>

if the sequence

$$(f_n(z))$$

is convergent for every $z \in \Omega$

In this case if we let

$$f(z) = \lim_{n \to \infty} f_n(z)$$

then we obtain a function $f : D \to \mathbb{C}$.

and we say that

$$f_n \to f$$

or $\lim_{n\to\infty} f_n(z) = f(z)$ pointwise

eg. If $\Omega = \mathbb{C}$, then the seq. (f_n) given by $f_n(z) = \frac{z}{n}$ is pointwise convergent and its limit is the zero function.

~~On the other hand~~

On the other hand the seq (g_n) defined by $g_n(z) = z^n$

is not convergent on $\Omega = \mathbb{C}$

eg $z=-1 \Rightarrow ((-1)^n)$ is not convergent). However if $\Omega = B(0,1) = \{z \in \mathbb{C} : |z|<1\}$ then (g_n) is pointwise convergent in Ω and the limit is the zero function.

<u>Def</u> : let $\Omega \subseteq \mathbb{C}$ and (f_n) be a sequence of functions on Ω. We say that (f_n) is uniformly convergent, if there is a function $f : \Omega \to \mathbb{C}$ with the property that for every $\varepsilon > 0$ $\exists\, n_0 \in \mathbb{N}$ ~~with the property that for every~~ ~~$\varepsilon>0$~~ such that

$$|f_n(z) - f(z)| < \varepsilon$$

$$\forall\, n > n_0 \text{ and } \forall\, z \in \Omega$$

In this case, we say that $f_n \to f$ uniformly or that f is the uniform limit of (f_n)

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Observe that if $f_n \to f$ uniformly then $f_n \to f$ pointwise

eg: let ~~$\Omega \in \mathbb{C}$~~ $\Omega = \mathbb{C}$ and consider the sequence (f_n) where $f_n(z) = \frac{z}{n}$ for $z \in \Omega$

If $f: \mathbb{C} \to \mathbb{C}$ is the zero fn given by $f(z)=0$ then we've seen that

$$f_n \to f \quad \text{pointwise.}$$

However, given $\varepsilon > 0$ we cannot choose any $n_0 \in \mathbb{N}$ such that

$$\frac{|z|}{n} = \left|\frac{z}{n} - 0\right| < \varepsilon \quad \forall n \geq n_0 \text{ and } \forall z \in \mathbb{C}$$

Thus $f_n \not\to f$ uniformly on $\Omega = \mathbb{C}$.

However, if Ω is a bounded subset of $\mathbb{C}$, then we can find $R>0$ such that $|z| \leq R$ $\forall z \in \Omega$ and now for a given $\varepsilon > 0$, if we choose $n_0 > \frac{R}{\varepsilon}$. then $\left|\frac{z}{n} - 0\right| \leq \frac{R}{n} \leq \frac{R}{n_0} < \varepsilon$ $\forall n > n_0$ and $\forall z \in \Omega$

(ii) $(g_n) = z^n$ on $\Omega = B(0,1)$

Then $g_n \to g$ pointwise when $g(z)=0$ $\forall z \in \Omega$ but $g_n \not\to g$ uniformly on $\Omega = B(0,1)$ (Check)

However $g_n \to g$ uniformly on $B(0,r)$ for any ~~$r<1$~~ fixed $0<r<1$

Property : If (f_n) is uniformly convergent sequence of functions on $\Omega \subseteq \mathbb{C}$ and $f = \lim_{n \to \infty} f_n$ then f_n is continuous on Ω $\forall n \Rightarrow f$ is continuous on Ω.

and,
If f_n is continuous for each n, and γ is a piecewise smooth path in Ω then

$$\int_\gamma f_n(z)\,dz \to \int_\gamma f(z)\,dz$$

$$\Rightarrow \lim_{n \to \infty} \int_\gamma f_n(z)\,dz \to \int_\gamma (\lim f_n(z))\,dz$$

proof :
(1) let $z_0 \in \Omega$. let $\varepsilon > 0$ be given

To show : $\exists\ \delta > 0$ such that

$$z \in \Omega \text{ and } |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$$

since $f_n \to f$ uniformly, $\exists\ m \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \frac{\varepsilon}{3} \quad \forall n > m \ \& \ \forall z \in \Omega.$$

In particular

$$|f_m(z) - f(z)| < \frac{\varepsilon}{3} \quad \text{and}$$

$$|f_m(z_0) - f(z_0)| < \frac{\varepsilon}{3}$$

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By the continuity of f_m at z_0, $\exists\ \delta > 0$ such that $z \in \Omega$ and $|z - z_0|\ \delta \Rightarrow |f_m(z_0) - f_m(z_0)| < \frac{\varepsilon}{3}$

Now $z \in \Omega$ and $|z - z_0| < \delta$

$$\Rightarrow |f(z) - f(z_0)| = |f(z) - f_m(z) + f_m(z) - f_m(z_0) + f_m(z_0) - f(z_0)|$$

$$\leq |f(z) - f_m(z)| + |f_m(z) - f_m(z_0)| + |f_m(z_0) - f(z_0)|$$

$$\cancel{<} = \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$\cancel{<} = \varepsilon$$

This prooves that f is continuous at z_0.

(ii) Let $L = \text{length}(\gamma) = \int_a^b |\gamma'(t)|\, dt.$

since $f_n \to f$ uniformly on Ω $\exists\ n_0 \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \frac{\varepsilon}{L+1} \quad \forall\ n > n_0 \in \mathbb{N}\ \& \ \forall\ z \in \Omega$$

now

$$\left| \int_\gamma f_n(z)\, dz - \int_\gamma f(z)\, dz \right|$$

$$\leq \int_\gamma |f_n(z) \cancel{dz} - f(z)|\, |dz|$$

$$\leq \left[\frac{\varepsilon}{L+1}\right] L < \varepsilon \quad \forall\ n > n_0 \text{ so } \int f_n \to \int f$$

eg. let $\Omega = B(0,1) \cup \{1\} = \{z \in \mathbb{C} : z = 1 \text{ or } |z| < 1\}$ and sequence on Ω is given by $f_n(z) = z^n$ for $z \in \Omega$. Then $f_n \to f$ pointwise where $f : \Omega \to \mathbb{C}$ is given by

$$f(z) = \begin{cases} 0 & \text{if } |z| < 1 \\ 1 & \text{if } |z| = 1 \end{cases}$$

each function $(f_n)_n$ is continuous on Ω but

f is not continuous on Ω

<u>Property</u> : let Ω be an open, connected subset of $\mathbb{C}$. If (f_n) is a sequence of analytic function on Ω and if $f_n \to f$ uniformly on Ω then f is analytic on Ω

<u>proof</u> let γ be any simple, closed, piecewise smooth path in Ω by Cauchy theorem

$$\int_\gamma f_n(z)\, dz = 0.$$

Hence by (ii) <u>proof</u> above <u>prop</u>.

$$\int_\gamma f(z)\, dz = 0$$

since this is true for any γ by Morera's theorem, f is analytic on Ω

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prop : Suppose (f_n) is sequence of functions on an open connected set Ω and $f_n \to f$ uniformly on Ω and each f_n is analytic on Ω, then

$$f_n^{(m)} \to f^{(m)} \quad \text{at every point on } \Omega$$

Moreover, the convergence is uniform on any ~~closed~~ disc $\overline{B}(z_0,r) \subseteq \Omega \Rightarrow \overline{B}(z_0,r) = \{z \in \mathbb{C} : |z-z_0| \leq r\} \subseteq \Omega$

proof : We have already seen that f is analytic on Ω. Hence $f^{(m)}$ exist on Ω.
Also, for any $z_0 \in \Omega$, if $r>0$ s.t. $\overline{B}(z_0,r) \subseteq \Omega$. then

$r<s<\rho<r$ $$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\omega)}{(\omega-z)^2}\,d\omega \quad \text{for any } z \in B(z_0,s)$$

where C is a circle $|\omega - z_0| = \rho$

Also $$f_n'(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\omega)}{(\omega-z)^2}\,d\omega$$

since $f_n \to f$ uniformly, given $\varepsilon>0$ we can find $m \in \mathbb{N}$ such that

$$|f_n(\omega) - f(\omega)| < \varepsilon \quad \forall\, n \geq m \;\; \& \; \forall\, \omega \in \Omega$$

Now,

$$|f_n'(z) - f'(z)|$$

$$\leq \frac{1}{2\pi}\int_C \frac{|f_n(\omega) - f(\omega)|\,|d\omega|}{|\omega - z|^2}$$

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Now for any ω such that $|\omega - z_0| = \rho$ and any $z \in B(z_0, S)$ we have

$$|\omega - z| \geq \rho - S$$

$$\frac{1}{|\omega - z|^2} \leq \frac{1}{|\rho - S|^2}$$

Thus.

$$\left|f_n'(z) - f'(z)\right| \leq \frac{\varepsilon \cdot 2\pi\rho}{(\rho-S)^2 \cdot 2\pi}$$

$$= \frac{\rho}{(\rho - S)^2}\varepsilon$$

Thus $f_n' \to f'$ uniformly on $B(z_0, S)$

similarly

$$f_n^{(m)} \to f^{(m)}$$

Series of functions :-

The series $\sum f_n(z)$ is pointwise convergent if the^n sequence $s_n(z)$ is piecewise convergent and uniformly convergent seq. $s_n(z)$ is uniformly convergent.
where $s_n(z) = \sum_{k=1}^{n} f_k(z)$

If $\sum f_n(z)$ is uniformly convergent then $f_n(z) \to 0$ uniformly. Uniformly Cauchy defined similarly and uniform Cauchy $\Rightarrow$ uniform convergent (for $\mathbb{C}$ valued function on $\Omega \subseteq \mathbb{C}$)

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Weierstrass M- Test :

If $\sum f_n(z)$ is a series of functions on Ω with the property that

$$|f_n(z)| \le M_n \qquad \forall\ z \in \Omega$$

where M_n is a sequence of non-negetive real numbers s.t. $\sum M_n$ is convergent, then $\sum f_n(z)$ is uniformly convergent.

proof let $S_n(z) = \sum_{k=1}^{n} f_k(z)$

for $n>m$ we have

$$|S_n(z) - S_m(z)| = \left| \sum_{k=m+1}^{n} f_k(z) \right|$$

$$\le \sum_{k=m+1}^{n} |f_k(z)|$$

$$\le \sum_{k=m+1}^{n} M_k \quad \forall\ z \in \Omega$$

Since $\sum M_k$ is convergent, it follows from above that $(S_n(z))$ is uniformly Cauchy and hence uniformly convergent on Ω