MA 205 Complex Analysis.

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lecture 12 Dated 25/05/09

f_n is a sequence of fns on $\Omega \subseteq \mathbb{C}$

$f_n \to f$ pointwise if $f_n(z) \to f(z)$ $\forall z \in \Omega$

$f_n \to f$ uniformly if $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$

such that $|f_n(z) - f(z)| < \varepsilon$ $\forall n \geq n_0$ and $\forall z \in \Omega$

uniform convergence $\Rightarrow$ pointwise convergence

but not conversely

Theorem.

If $f_n \to f$ uniformly on Ω then

(i) f_n continuous $\forall n \Longrightarrow f$ continuous.

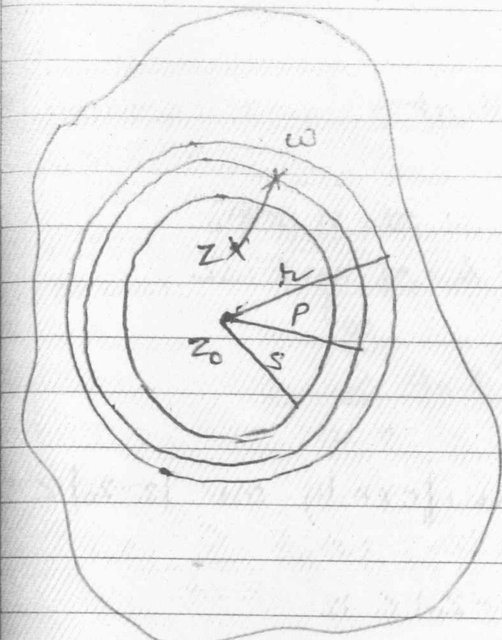
(ii) If f_n is continuous; $\forall n$ and γ is a piecewise smooth path in Ω then

$$\int_\gamma f_n(z)\,dz \longrightarrow \int_\gamma f(z)\,dz$$

(iii) If Ω is open connected and each f_n is analytic on Ω, then f is analytic on Ω

(iv) If Ω is open, connected and contains a closed disc $D = \{z \in \mathbb{C} : |z - z_0| \leq r\}$ and f_n is analytic on Ω for each n, then

$f_n^{(m)} \longrightarrow f^{(m)}$ uniformly on the open disc $|z-z_0| \leq r$ for $s < r$



Recall the idea of proof of (iv):

Choose ρ such that $s < \rho < r$ for $z \in \mathbb{C}$ with $|z - z_0| < s$

$$f_n^{(m)}(z) = \frac{m!}{2\pi i} \int \frac{f_n(w)\, dw}{(w-z)^{m+1}}$$

where $|w - z_0| = \rho$

$$f^{m}(z) = \frac{m!}{2\pi i} \int \frac{f(w)\, dw}{(w-z)^{m+1}}$$

$$f_n^{(m)}(z) - f^{(m)}(z) = \frac{m!}{2\pi i} \int_{|w-z_0|=\rho} \frac{f_n(w) - f(w)}{(w-z)^{m+1}}\, dz$$

Now $|w - z| \geq \rho - s$ $\forall$ w on the circle $|w - z_0| = \rho$ and $\forall$ z such that $|z - z_0| < s$

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and so

$$\left| f_m^{(m)}(z) - f^{m}(z) \right| \leq \frac{m!}{2\pi} \frac{1}{(\rho - s)^{m+1}} \int \left| f_n(\omega) - f(\omega) \right| |d\omega|$$

$$\leq \frac{m!\,(2\pi\rho)}{2\pi\,(\rho-s)^{m+1}} \, \varepsilon \qquad \forall\, n \geq n_0 .$$

where $n_0 \in \mathbb{N}$ such that

$$|f_n(\omega) - f(\omega)| < \varepsilon \qquad \forall\, n > n_0 \;\; \& \;\; \forall\, \omega \in \Omega$$

This implies that

$$f_n^{(m)} \to f^{(m)} \quad \text{uniformly on } |z - z_0| < s$$

on the open disc $|z - z_0| < r$

Weierstrass M - Test :

If $\sum_n f_n(z)$ is a series of complex valued functions on $\Omega \subseteq \mathbb{C}$ and M_n is a sequence of non-negetive real nos such that

(i) $|f_n(z)| \leq M_n \quad \forall\, n \in \mathbb{N}$ and $\forall\, z \in \Omega$
(ii) $\sum M_n$ is convergent

Then $\sum f_n(z)$ is uniformly convergent on Ω

proof $\left| \sum_{k=m+1}^{n} f_k(z) \right| \le \sum_{k=m+1}^{n} |f_k(z)| \le \sum_{k=m+1}^{n} M_k$

POWER SERIES :

By power series we mean a series of functions of the form

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{where } a_n \in \mathbb{C}$$

(i.e. $\sum_{n=0}^{\infty} f_n(z)$ where $f_n(z) = a_n z^n$)

eg : The geometric series

$$\sum_{n=0}^{\infty} z^n$$

is one example of power series that this series

- converges absolutely in the open disc $|z|<1$
- converges uniformly in the disc $|z| \le r$ for any $r<1$
- diverges if $|z| \ge 1$ since $|z^n| \ge 1$ and so $z^n \not\to 0$

General case :

If $\sum a_n z^n$ is any power series, then we define

$R = \sup \{ r \in \mathbb{R} : (|a_n| r^n \text{ is bounded} \}$
provided the set $\{ r \leq \mathbb{R} : (|a_n| r^n \text{ is bounded} \}$ is a bounded subset of $\mathbb{R}$ (by the completness property of $\mathbb{R}$ and since $r = 0$ is always an element of this set, we see that the set has the supremum if it is bounded)

For a subset S of $\mathbb{R}$ a supremum of S is a real number M such that

(i) M is an <u>upper bound</u> of S, i.e.,

$$x < M \ \forall \ x \in S$$

(ii) If $\alpha \in \mathbb{R}$ is any upper bound of S then $M \leq \alpha$

equivalently,
for every $\epsilon > 0$ $\exists$ $x \in S$
such that

$$M - \varepsilon < x \leq M$$

<u>Note</u> : If a set S has a supremum, then it is unique, then it is unique and denoted by $\sup S$.

FACT : Every nonempty bounded subset of $\mathbb{R}$ has a supremum (and and "infimum") in $\mathbb{R}$ (Completeness property of $\mathbb{R}$)

eg : $S = \{ x = \mathbb{Q} : x^2 < 2 \}$ is a bounded subset of $\mathbb{Q}$ and $\sup S = \sqrt{2}$. However S does not have a sf supremum in $\mathbb{Q}$

We set $R = \infty$ if the set $\{ r \in \mathbb{R} : |a_n| r^n$ is bounded $\}$ is unbounded. R is called the radius of convergence of the power series

$$\sum a_n z^n$$

Prop : If R is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$, then

(i) $\sum a_n z^n$ converges absolutely in the open disc $|z| < R$

(ii) $\sum a_n z^n$ diverges at every $z \in \mathbb{C}$ such that $|z| > R$.

(iii) $\sum a_n z^n$ converges uniformly in the closed disc $|z| \leq S$ for any $S < R$

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proof

(i) let $z \in \mathbb{C}$ be such that $|z| < R$

By definition of supremum, $\exists\, r \in \mathbb{R}$ such that

$$|z| < r \leq R$$

and $(|a_n| z^n)$ is bounded

Suppose

$$|a_n| r^n < \alpha \quad \forall\, n \in \mathbb{N}$$

$$0 < \sum |a_n z^n| = \sum |a_n| |z^n|$$

$$= \sum (a_n z^n) \left(\frac{|z|}{r}\right)^n$$

$$\leq \alpha \left(\frac{|z|}{r}\right)^n$$

Since $\frac{|z|}{r}$ is less than 1 the series $\sum \left(\frac{|z|}{r}\right)^n$ is convergent so by comparison test $\sum |a_n z^n|$ is convergent, i.e. $\sum a_n z^n$ is absolute convergent. This prooves (i)

(ii) If $|z| > R$ then

$|a_n z^n| = |a^n||z^n|$ is unbounded and

hence $|a_n z^n| \not\to 0$

$\Rightarrow a_n z^n \not\to 0$

$\Rightarrow \sum a_n z^n$ is divergent

this prooves (ii)

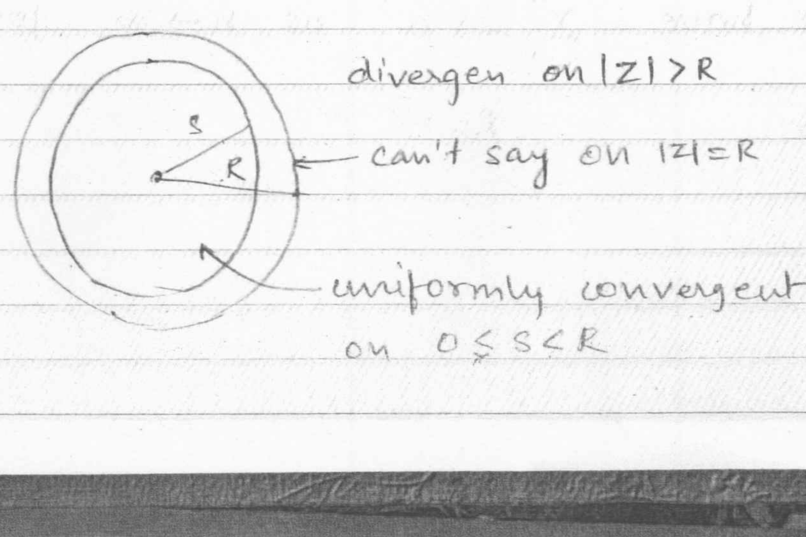
(iii) If $s < R$ then theore $r \in \mathbb{R}$ such that $s < r \le R$ and $|a_n| r^n$ is bounded, Hence as in (i) $\sum |a_n| s^n$ is convergent

Also $|z| \le s$

$\Rightarrow |a_n z^n| \le |a_n| s^n$

Hence by Weiesstrass M-test,

$\sum a_n z^n$ is uniformly convergent for $|z| \le s$



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Remark: More generally series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{where } a_n \in \mathbb{C}$$

is called a power series (around a point z_0) The radius of convergence of such a power series is defined in the same way.

$$R = \begin{cases} \sup D & \text{if } D \text{ is bounded} \\ \infty & \text{if } D \text{ is unbounded} \end{cases}$$

where $D = \{ r \in \mathbb{R} : (|a_n| r^n \text{ is bounded} \}$

we have result analogous to above proposition with z replaced by $(z-z_0)$

eg $$\sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad a_n = \frac{1}{n!}$$

we have $\frac{r^n}{n!} \to 0$ as $n \to \infty$ for any $r \in \mathbb{R}$

so $R = \infty$