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Lecture 13. Dated 25/05/09

Tests for determining the radius of convergence.

let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ be a power series

(i) Ratio test : If the limit

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exist, then this limit is ~~known~~ the radius of convergence.

(ii) Root test : If the limit

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

exist, then the radius of convergence R is

$$R = \begin{cases} 1/L & \text{if } L \neq 0 \\ \infty & \text{if } L = 0 \end{cases}$$

proof

(i) Let R be the radius of convergence and $\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

To show : $\rho = \sup \{ r \in \mathbb{R} : |a_n| r^n \text{ is bounded} \}$

Let $r \in \mathbb{R}$ be such that $(|a_n| r^n$ is bounded

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<u>claim 1</u>: $R \leq \rho$

suppose $r > \rho$. Then $\exists\, n_0 \in \mathbb{N}$ such that

$$\left|\frac{a_n}{a_{n+1}}\right| < r \qquad \forall\, n \geq n_0$$

$$|a_n| < r\,|a_{n+1}| \qquad \forall\, n \geq n_0$$

$$\Rightarrow |a_{n_0}| < r\,|a_{n_0+1}| < r^2\,|a_{n_0+2}| < \ldots < r^k |a_{n_0+k}|$$

$$\forall\, k \geq 1$$

$$\Rightarrow \quad r^{n_0+k}\,|a_{n_0+k}| > r^{n_0}\,|a_{n_0}| > 0 \qquad \forall\, n \geq n_0$$

$$\Rightarrow \quad r^n\,|a_n| > r^{n_0}\,|a_{n_0}| > 0 \qquad \forall\, n \geq n_0$$

$$\Rightarrow \quad r^n\,|a_n| \not\to 0 \Rightarrow \sum a_n (z-z_0)^n \text{ is not convergent if } |z-z_0| \geq r$$

Since the n^{th} term

$$a_n |z-z_0|^n \geq a_n r^n \not\to 0$$

hence $R \leq r$

Thus $R \leq r$ for any $r > \rho$

hence $R \leq \rho$ —(1)

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Claim 2 $\quad R \geq \rho \qquad$ If $\rho = 0$, then clearly $R \geq 0$
If suppose $\rho > 0$

let $r \in \mathbb{R}$ be such that $0 < r < \rho$

$$\left|\frac{a_n}{a_{n+1}}\right| > r \qquad \forall\ n \geq n_0$$

$$|a_{n_0}| > r\,|a_{n_0+1}| \qquad \text{for } n = n_0$$

$$> r^2\,|a_{n_0+2}| \ \ldots$$

$$|a_{n_0}| > r^k\,|a_{n_0+k}|$$

or $\qquad r^n |a_n| < |a_{n_0}|\, r^{n_0} \qquad \forall\ n \geq n_0$

Hence for all $n > n_0$ ~~are~~, $|a_n| r^n$ is bounded.

and so.

$$r \leq R$$

Since this holds for every $r < \rho$ we must have

$$\rho \leq R \qquad \text{— (2)}$$

So $\rho = R$ is prooved from (1) and (2) (prooved)

proof

(ii) Case I $\quad L \neq 0 \qquad$ then $L > 0$

~~claim~~ claim: $\quad R \leq \frac{1}{L}$

let $r > \frac{1}{L}$ i.e. $\frac{1}{r} < L$

Then $\exists\ n_0 \in \mathbb{N}$ such that

$$\frac{1}{r} < \sqrt[n]{|a_n|} \qquad \forall\ n \geq n_0$$

$\Rightarrow$ $r^n |a_n| > 1 \qquad \forall\ n \geq n_0$

$\Rightarrow$ $r^n |a_n| \not\to 0$

$\Rightarrow$ $\sum a_n(z-z_0)^n$ is not convergent if $|z-z_0| \geq r$

$\Rightarrow$ $R \leq r$

Since this holds for any

$r > \frac{1}{L}$ we obtain

$$R \leq \frac{1}{L}$$

<u>Claim 2</u> : $R \geq 1/L$

<u>proof</u> let $r \in \mathbb{R}$ be such that

$$0 < r < \frac{1}{L}$$

$$L < \frac{1}{r}$$

$\exists$ some $n_0 \in \mathbb{N}$ such that

$$\sqrt[n]{|a_n|} < \frac{1}{r}$$

$$r^n \left(\sqrt[n]{|a_n|}\right)^n < 1$$

$$r^n |a_n| < 1 \qquad \forall\ n > n_0$$

$\Rightarrow$ $r^n |a_n|$ is bounded

$\Rightarrow$ $r \leq R$ $\qquad \forall\ r \geq 0$

Since this holds for every $r < \frac{1}{L}$ we must have

$\Rightarrow$ $R \geq \frac{1}{L}$

Thus $R = \frac{1}{L}$

Finally when $L = 0$, then as in case 2 for any $r > 0$, we see that $\exists\ n_0 \in \mathbb{N}$ such that

$$\sqrt[n]{|a_n|} < \frac{1}{r} \qquad \forall\ n > n_0$$

$\Rightarrow$ $r^n |a_n|$ is bounded

So $D = \{ r \in \mathbb{R} : (|a_n| r^n \text{ is bounded} \} \supseteq (0, \infty)$

eg

(i) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\qquad$ Here $a_n = \frac{1}{n!}$

$$\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1/n!}{1/(n+1)!} = n+1 \to \infty$$

as $n \to \infty$

So $R = \infty$

(ii) $\sum n^n z^n$

here $|a_n| = |n^n|$ $\Rightarrow$ $\sqrt{|a_n|} = n \to \infty$

as $n \to \infty$

So $R = 0$.

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Remark : A general formula for the radius of convergence of a power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

is

$$R = \begin{cases} 1/L & \text{if } L \neq 0,\ L \neq \infty \\ 0 & \text{if } L = \infty \\ \infty & \text{if } L = 0 \end{cases}$$

where $L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

Thus

$$\boxed{R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}}$$

Cauchy-Hadamard formula.

Def : The limsup of a sequence of $\{x_n\}$ of real numbers is defined as follows

$$\limsup_{n\to\infty} x_n = \begin{cases} \lim_{n\to\infty} M_n & \text{if } (x_n) \text{ is bounded} \\ \infty & \text{if } (x_n) \text{ is unbounded.} \end{cases}$$

where

$$M_n = \sup \{x_n, x_{n+1}, \ldots\}.$$

$$M_1 \geq M_2 \geq \ldots \geq 0.$$

also $\lim M_n$ exist

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In case $\lim_{n\to\infty} x_n$ exist then

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} x_n$$

However limsup always exists in $\mathbb{R} \cup \{\infty\}$ for any sequence (x_n) of non negetive real numbers.

exercise: proove the Cauchy-Hadamard formula (hint) Use similar argument as in the pf of the root test

Relation of power series with analytic function

Theorem: Let

$\sum a_n z^n$ be a power series with radius of convergence $R > 0$. Then the function

$f : D \to \mathbb{C}$ where $D = \{z \in \mathbb{C} : |z| < R\}$

defined by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is analytic on D and $f'(z)$

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

$$f''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$$

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$$f^{(m)}(z) = \sum_{n=m}^{\infty} n(n-1)\ldots(n-m+1)\, z^{n-m}$$

for every $z \in D$.

proof Since R is the radius of convergence, f is well defined. Further if

$$f_n(z) = \sum_{k=0}^{n} a_k z^k$$

then each f_n is analytic on D (in fact entire being a polynomial) and
Then

$$f_n(z) \longrightarrow f(z) \quad \text{in } D$$

and convergence is uniform in a closed disk

$$\bar{D}_r = \{ z \in \mathbb{C} : |z| \leq r \} \qquad \forall\ r < R$$

Hence by results on uniform convergence $f(z)$ is analytic on D_r for every $r < R$
and

$$f_n^m(z) \longrightarrow f^m(z)$$

on

$$D_s = \{ z \in \mathbb{C} : |z| < s \}$$

for every $s < r$

Thus f is analytic on D and

$$f^m(z) = \lim_{n \to \infty} f_n^m(z)$$

$$= \lim_{n \to \infty} \sum_{k=m}^{n} a_k k(k-1)\ldots(k-m+1)\, z^{k-m}$$

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$$= \sum_{k=m}^{\infty} k(k-1)\ldots\ldots(k-m+1)\, a_k z^{k-m}$$

as desired

In particular

$$\boxed{a_m = \frac{f^m(0)}{m!}}$$

terms for Taylor series expansion.

Theorem 2 : Suppose $f : \Omega \to \mathbb{C}$ is analytic at an interior point z_0 of Ω. Then f is analytic on $|z-z_0| < r$ for some $r>0$ and f is given by a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{for } |z-z_0| < r$$

where the power series has radius of convergence $\geq r$. For any fixed $s \in \mathbb{R}$ with $0<s<r$ we have

$$a_n = \frac{1}{2\pi i}\int \frac{f(w)}{(w-z_0)^{n+1}}\, dw.$$

where c denotes the circle $|z-z_0| = s$

In particular if $|f(w)| \leq M$ on c then

$$|a_n| \leq \frac{M}{s^n}$$