MA 205 Complex Analysis

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lecture 14 Dated 26-05-09

proof of theorem 2.

let $s \in \mathbb{R}$ be such that $0 < s < r$
Assume $z_0 = 0$ for simplicity

Choose any $z \in B(z_0, s)$ or $B(0, s)$. Then by cauchy Integral formula

$$f(z) = \frac{1}{2\pi i} \int \frac{f(w)\, dw}{(w - z)}$$

$$\frac{1}{w-z} = \frac{1}{w\left[1 - \frac{z}{w}\right]} = \frac{1}{w}\left[1 + \frac{z}{w} + \frac{z^2}{w^2} + \cdots\right]$$

which is valid since $\left|\frac{z}{w}\right| = \frac{|z|}{w} < 1$

Thus $$\frac{f(w)}{w-z} = \sum_{n=0}^{\infty} \frac{f(w)\, z^n}{w^{n+1}}$$

and the series in right converges uniformly for any w with $|w| = s$ (for every fixed $z \in B(0,s)$)
Hence

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)\, dz}{(w-z)} = \sum_{n=0}^{\infty} \frac{z^n}{2\pi i} \int \frac{f(w)\, dw}{w^{n+1}}$$

$$= \sum_{n=0}^{\infty} a_n z^n \quad \text{where } a_n = \frac{1}{2\pi i} \int \frac{f(w)}{w^{n+1}}\, dw$$

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Thus for every $z \in B(0,s)$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where the series in right is convergent.

$\because$ s is any real number such that $0 < s < r$ we obtain

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and by theorem 1, $a_m = \dfrac{f^{m}(0)}{m!}$

Finally if $|f(w)| \leq M$ on C then

$$|a_n| = \left|\frac{1}{2\pi i}\int \frac{f(w)\,dw}{w^{n+1}}\right|$$

$$= \frac{1}{2\pi}\int \frac{|f(w)|\,|dw|}{|w^{n+1}|} \leq \frac{M.\cancel{2\pi}s}{\cancel{2\pi}\, s^{n+1}}$$

$$= \frac{M}{s^n}$$

<u>exercise</u> : rewrite the above proof without assuming $z_0 = 0$.

<u>remark</u> : The above theorem shows that if f is analytic at z_0, then the

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Taylor series of f at z_0, namely

$$\sum_{m=0}^{\infty} \frac{f^{m}(z_0)}{m!} (z-z_0)^m$$

is convergent in a ball ~~B(z~~, $B(z_0, r)$ of radius $r>0$ and it converges to $f(z)$ $\forall$ $z \in B(z_0,r)$

A similar result is not true for functions of real variables, for example consider

$$e^{-1/x^2} = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \frac{e^{-1/x^2}}{x} \to 0$$

In fact $\frac{e^{-1/x^2}}{x^m} \to 0$ for any $m \geq 0$

and using this we see that

$f^{m}(0)$ exist and $f^{m}(0) = 0$ $\forall$ $m \geq 0$

Thus f is infinitely differentiable on $\mathbb{R}$

However the taylor series of f at 0

$$\sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} x^m$$

is a zero function hence it is not equal to $f(x)$ for any $x \neq 0$.

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Corollary : If f is analytic at z_0 and

$f^{(m)}(z_0) = 0 \qquad \forall\ m \geq 0$

then there is $r > 0$ such that

$f(z) = 0$ for all $z \in B(z_0, r)$

Theorem : The zeroes of a non zero analytic function are isolated i.e. if $f : \Omega \to \mathbb{C}$ is analytic at and interior point z_0 of Ω and ~~$f(z_0) = 0$, then $\exists\, r > 0$ such that $f(z) \neq 0\ \forall\, z \in \Omega$ such that $0 < |z - z_0| < r$~~ and non zero on $B(z_0, r)$ with $f(z_0) = 0$ then $\exists\ \delta > 0$ such that $f(z) \neq 0\ \forall\ z \in \Omega$ such that $0 < |z - z_0| < \delta$

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Corollary (Identity principle) If Ω is an open, connected subset of $\mathbb{C}$ and $f, g : \Omega \to \mathbb{C}$ are analytic function such that

$$f(z) = g(z) \qquad \forall z \in S$$

where $S \subseteq \Omega$ having a limit point (i.e. $\exists z_0 \in \Omega$ such that every $B(z_0, \epsilon)$ contains a point of S other than z_0)

then $$f(z) = g(z) \qquad \forall z \in \Omega$$

proof let $h = f - g$ and z_0 be a limit point of S. Then $z_0 \in \Omega$ and there is a sequence (z_n) of points of S such that

$$z_n \to z_0$$

(eg z_n can be chosen as a point of S in $B(z_0, \frac{1}{n})$ other than z_0. Now f, g are analytic and hence continuous. So.

$$f(z_n) \to f(z_0) \text{ and } g(z_n) \to g(z_0)$$

$$\Rightarrow h(z_n) \to h(z_0)$$

But $h(z_n) = 0 \ \forall n$ since $z_n \in S$.

So. $h(z_0) = 0$ So z_0 is a ~~zo~~ zero of the analytic for h but since z_0 is a limit point of S, it is not an isolated zero of h, which contradicts previous theorem unless

$h = 0$, i.e $f = g$.