MA 205 Complex Analysis

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lecture 15. Dated 20/05/09

Proposition : Zeros of a non zero analytic function are isolated, i.e. if f is analytic on $B(z_0, r)$ for some $z_0 \in \mathbb{C}$ and $r>0$ and $f(z_0)=0$, but f is not identically zero on $B(z_0, r)$, then $\exists\, \delta>0$ such that

$f(z) \neq 0 \quad \forall\ z \in B(z_0, r)$ such that $0<|z-z_0|<\delta$

proof : Let $D = B(z_0, r)$. Note that D is open and connected. We'll first show that

Claim : z_0 is a zero of f of finite order i.e. $\exists\ m \geq 1$ such that

$$f^{(k)}(z_0) = 0 \quad \forall\ 0 \leq k < m \quad \text{but } f^{m}(z_0) \neq 0$$

let $U = \{ w \in D : f^{(m)}(w) = 0 \quad \forall\ m \geq 0 \}$

and $V = D \setminus V = \{ w \in D : w \notin U \}$

Clearly U and V are disjoint, and

$$D = U \cup V.$$

Also since f is not identically zero on D, the V is non empty.

Further, let $w \in U$ since f is analytic on D, we have

$$f(w) = \sum_{n=0}^{\infty} a_n (w-w_0)^n \quad \text{on} \quad |w-w_0| < s$$

for some $s > 0$. But $a_n = \frac{f^n(w_0)}{n!} = 0 \ \forall \ n \geq 0$ and so $f(w) = 0 \ \forall \ w \in B(w_0, s)$ and hence $f^m(w) = 0 \ \forall \ w \in B(w_0, s)$. Thus $B(w_0, s) \subseteq U$.

This prooves that U is open.

Also if $w_0 \in V$, then $f^m(w_0) \neq 0$ for some $m \geq 0$. Since $f^{(m)}$ is continuous on D $\exists \ \rho > 0$ such that $f^m(w) \neq 0 \ \ \forall w \in B(w_0, \rho)$. Thus $B(w_0, \rho) \subseteq V$.

This prooves V is open. Thus if V were non-empty, then we'll obtain that D is not connected which is a contradiction. So we must have $U = \phi$. In other words every zero of f in D is of finite order. This prooves the claim.

To complete the proof, suppose $m \geq 1$ is such that

$$f^m(z_0) \neq 0, \ \text{but} \ f^{(k)}(z_0) = 0 \quad \forall \ k < m$$

Now by the continuity of f^m, $\exists \ \delta > 0$ such that

$$f^m(z) \neq 0 \qquad \forall z \in B(z_0, \delta)$$

So, the power series expansion of f around z_0,

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$\therefore$ $$f(z) = \sum_{k=0}^{\infty} \frac{f^{k}(z_0)}{k!}(z-z_0)^k$$

$$= (z-z_0)^m h(z)$$

which is valid in $B(z_0, s)$ for some $s>0$ and

$$h(z) = \frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \cdots$$

Now $h(z)$ is analytic in $B(z_0, s)$ and $h(z_0) \neq 0$

Since h is continuous, being analytic, $\exists\ \delta > 0$ such that

$h(z) \neq 0 \quad \forall\ z \in D$ such that $|z-z_0| < \delta$

But $f(z) = (z-z_0)^m h(z)$

and so

$f(z) \neq 0$ for $0 < |z-z_0| < \delta$ which prooves the proposition

Remark : The above proof remains valid if D is open, connected set rather than a disc. As a result we obtain the Identity principle as a consequence of the above proposition (as shown already)

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Zeros of analytic function :

Let Ω be open, connected subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be analytic function that is not identically zero on Ω.

Then as seen in the proof of above proposition, every zero of f on Ω is of finite order, i.e., if $f(z_0) = 0$ then there is $m \in \mathbb{N}$ such that

$$f^{m}(z_0) \neq 0 \quad , \quad \text{but} \quad f^{(k)}(z_0) = 0 \quad \forall \; k < m$$

or equivalently

$$f(z) = (z - z_0)^m h(z)$$

in an open ball around z_0, with $h(z)$ analytic and $h(z_0) \neq 0$. In this case we say that z_0 is a zero of f of order m. when $m = 1$, z_0 is called a simple zero. when $m = 2$, z_0 is called a double zero.

eg. (i) $f(z) = (z-1)^3 (z-2) z^5 e^z$

$z = 0, 1, 2$ of order $5, 3, 1$ respectively

(ii) $f(z) = \sin z$

$z = n\pi$ is a simple zero of f, for every $n \in \mathbb{Z}$

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eg. (1) $f(z) = e^z$

f is analytic on $\mathbb{C}$; so by Thm 2 f has a convergent power series expn.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where $a_n = \frac{f^{n}(0)}{n!} = \frac{1}{n!}$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{for } z \in \mathbb{C}$$

In a similar way

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - - - -$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!}$$

and the power series expansion is valid over $\mathbb{C}$

Analyticity at ∞

Def: A function $f : \mathbb{C} \to \mathbb{C}$ is said to be analytic at ∞ if

$$\lim_{z \to \infty} f(z) = \lambda$$

for some $\lambda \in \mathbb{C}$ and the function $g: \mathbb{C} \to \mathbb{C}$ defined by

$$g(\omega) = \begin{cases} f(\frac{1}{\omega}) & \text{if } \omega \neq 0 \\ \lambda & \text{if } \omega = 0 \end{cases}$$

is analytic at $\omega = 0$.

In this case, g has a convergent power series expansion

$$g(\omega) = b_0 + b_1\omega + b_2\omega^2 + \cdots$$

Valid for $|\omega| < r$ for some $r > 0$ and hence we have

$$f(z) = b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots \qquad \text{for } |z| > r$$

In case $\lambda = b_0 = g(0) = 0$, we say that f has a zero at ∞, and the order of f at ∞ is the order of $\omega = 0$ as a zero of $g(\omega)$. Note that this order $= m$ if

$$f(z) = \frac{h(z)}{z^m} \qquad \text{for } |z| > r$$

where h is analytic at ∞ and h is non-zero at ∞

eg. $f(z) = \frac{1}{z^3}$ has a zero of order 3 at ∞

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ISOLATED SINGULARITIES :

Def A complex-valued function f is said to have an isolated singularity at $z_0 \in \mathbb{C}$ if f is analytic on $0<|z-z_0|<r$ for some $r>0$

We'll say f has an isolated singularity at ∞ if f is analytic on $|z| \geq R$ for some $R>0$

eg. $f(z) = \frac{1}{z}$ for $z \neq 0$

has an isolated singularity at $z=0$

$f(z) = \frac{\sin z}{z}$ for $z \neq 0$

has an isolated singularity at $z=0$ (removable singularity)

Def A isolated singularity z_0 of a function f is called a removable singularity if f can be extended to an analytic function on $B(z_0, r)$ for some $r>0$ otherwise it is called a (non removable singularity)

Riemann's theorem on Removable singularity:

If f has an isolated singularity then this is a removable singularity if and only if f is bounded in a punctured disc $0<|z-z_0|<r$ for some $r>0$