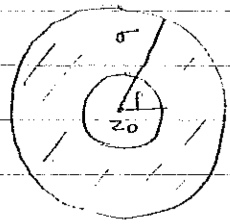
Lecture 16 Dated : 27/05/09

Theorem (Laurent Decomposition)

Suppose $\rho, \sigma \in \mathbb{R}$ with $0 \le \rho < \sigma$ and A be the annulus

$$A = \{ z \in \mathbb{C} : \rho < |z - z_0| < \sigma \}$$



around $z_0 \in \mathbb{C}$. If f is analytic on A, then we can write

$$f(z) = f_0(z) + f_1(z) \quad \forall z \in A$$

where f_0 is analytic in the disc $|z - z_0| < \sigma$ and f_1 is analytic on $|z - z_0| \ge \rho$. Moreover we can arrange that $f_1(\infty) = \lim_{z \to \infty} f_1(z) = 0$

and in this case f_0 and f_1 are uniquely determined by f

Proof : Uniqueness of f.

Suppose we have two different decompositions

$$f(z) = f_0(z) + f_1(z) = g_0(z) + g_1(z) \quad \forall z \in A \quad (*)$$

where, f_0 and g_0 are analytic in $|z - z_0| < \sigma$ and f_1 and g_1 are analytic in $|z - z_0| \ge \rho$ and at ∞, and $f_1(\infty) = g_1(\infty) = 0$.

To show : $f_0(z) = g_0(z) \quad \forall z$ with $|z - z_0| < \sigma$

$f_1(z) = g_1(z) \quad \forall z$ with $|z - z_0| > \rho$

Consider $h : \mathbb{C} \to \mathbb{C}$ defined by

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$$h(z) = \begin{cases} f_0(z) - g_0(z) & \text{if } |z-z_0| < \sigma \\ -f_1(z) + g_1(z) & \text{if } |z-z_0| > \rho \end{cases}$$

Note that h is well defined since by

$$f_0(z) - g_0(z) = g_1(z) - f_1(z)$$

on the overlap i.e. for $z \in A$. Also h is analytic on $\mathbb{C}$ and also at ∞ and

$$h(z) \to 0 \quad \text{as } z \to \infty$$

Consequently, h is a bounded function on $\mathbb{C}$. So by Liouville's theorem h is a constant function. Since $h(\infty) = 0$ we must have

$$h(z) = 0 \qquad \forall z \in \mathbb{C}$$

Thus $f_0(z) = g_0(z)$ $\forall z \in \mathbb{C}$ with $|z-z_0| < \sigma$

$f_1(z) = g_1(z)$ $\forall z \in \mathbb{C}$ with $|z-z_0| > \rho$

and $f_1(\infty) = g_1(\infty) = 0$

This prooves the uniqueness.

proof for Existence : Take any $r, s \in \mathbb{R}$ such that $0 < \rho < r < s < \sigma$, let

C_r be a circle $|z-z_0| = r$, and

C_s be a circle $|z-z_0| = s$

and $C = C_s - C_r$

be the boundary of the inner annulus

$$r < |z-z_0| < s$$

Now f is analytic on and inside the region bounded by C. Hence by Cauchy Integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{(\omega - z)} \, d\omega \quad \text{for } r < |z-z_0| < s$$

Thus. $$f(z) = \frac{1}{2\pi i}\int_{C_s} \frac{f(\omega)}{\omega - z}\, dz - \frac{1}{2\pi i}\int_{C_r} \frac{f(\omega)}{\omega - z}\, d\omega.$$

$$f_0(z) = \frac{1}{2\pi i}\int_{C_s} \frac{f(\omega)\, d\omega}{\omega - z} \quad \text{for } |z-z_0| < s$$

$$f_1(z) = \frac{-1}{2\pi i}\int_{C_r} \frac{f(\omega)\, d\omega}{\omega - z} \quad \text{for } |z-z_0| > r$$

Then, f_0 is analytic on $|z-z_0| < s$
f_1 is analytic on $|z-z_0| > r$

Also
$$f_1(z) \to 0 \text{ as } z \to \infty$$
and $f(z) = f_0(z) + f_1(z)$ for $r < |z-z_0| < s$

technically the functions f_0, f_1 depend on r and s. However, thanks to uniqueness of f_0 and f_1 prooved earlier, we see that by varying r, s, we obtain f_0 and f_1 that are analytic on $|z-z_0| < \sigma$ and on $|z-z_0| > \rho$ and at ∞, respectively, such that $f = f_0 + f_1$ on A and $f_1(\infty) = 0$. this prooves the theorem.

let f be analytic on $\rho < |z-z_0| < \sigma$ and
$$f(z) = f_0(z) + f_1(z)$$
be its laurant decomposition. Then f_0 analytic on $|z-z_0| < \sigma$

$$\Rightarrow f_0(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{for } |z-z_0| < \sigma$$

for uniquely determined a_n that are given by

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$$a_n = \frac{f_0^n(z_0)}{n!}$$

$$= \frac{1}{2\pi i}\int_{C_s} \frac{f_0(\omega)}{(\omega-z)^{n+1}}\,d\omega$$

for any $s \in \mathbb{R}$. Also

$f_1(z)$ is analytic in $|z-z_0| > \rho$ & ∞

$$\Rightarrow f_1(z) = b_0 + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} - - - - -$$

valid for $|z-z_0| > r$ for some $r > \rho$ with $b_0 = 0$.
Moreover since f_1 is analytic on $|z-z_0| > \rho$ the series

$$b_0 + b_1\omega + b_2\omega^2 + \ldots$$

has radius of convergence $\geq \frac{1}{\rho}$ and

$$f_1(z) = \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \ldots$$

Also

$$b_n = \frac{1}{2\pi i}\int \frac{g_1(\omega)}{\omega^{n+1}}\,d\omega = \frac{1}{2\pi i}\int \frac{f_1(z)}{(z-z_0)^{-(n+1)}}\,dz$$

exercise: Show that the radius of convergence of the power series

$$h(z) = \sum c_n (z-z_0)^n$$

is the largest R such that $h(z)$ extends to an analytic function on $|z-z_0| < R$. (Hint Use Thm 1 and 2)

Thus we have doubly infinite series expansion

$$f(z) = \ldots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 \ldots$$

valid for $\rho < |z-z_0| < r$

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or $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$. This is called the laurent series decomposition for f.

The doubly infinite series is absolutely convgnt on $\rho < |z-z_0| < \sigma$ and uniformly convergent on $r < |z-z_0| < s$ for any r, s with $\rho < r < s < \sigma$

Also $\displaystyle\int \frac{f(z)\,dz}{(z-z_0)^{m+1}} = \sum_{n=-\infty}^{\infty} \int \frac{a_n}{(z-z_0)^{m-n+1}}\,dz$

$$\int_{C_r} z^k\,dz = \begin{cases} 0 & k \neq -1 \\ 2\pi i & k = -1 \end{cases} \quad k = -1, -2, -3 \ldots$$

$$= \sum_{-\infty}^{\infty} \int a_n (z-z_0)^{-m+n-1}\,dz = 2\pi i\, a_m$$

Thus

$$a_m = \frac{1}{2\pi i} \int \frac{f(z)}{(z-z_0)^{m+1}}\,dz \quad \forall\, m \in \mathbb{Z}$$

In particular

$$|a_m| \leq \frac{M}{r^m} \quad \text{if } |f(z)| < M \text{ in } C_r$$

proof of Riemann's theorem on removable sing.

One direction is easy, i.e., if z_0 is a removable singularity of f then f extends to an analytic function on $|z-z_0| \leq r$ for some $r>0$ and this function is bounded on $|z-z_0| \leq r$ being a continuous function on a closed and bounded set. Hence f is bounded on $0 < |z-z_0| < r$

Conversely suppose f is bounded and analytic on $0 < |z-z_0| < \rho$ for some $\rho > 0$. Then $|f(z)| \leq M$ $\forall\, z \in \mathbb{C}$ with $0 < |z-z_0| < \rho$

Note f has a laurent series expansion

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n \quad \text{for } 0 < |z-z_0| < \rho$$

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where the co-eff a_n have the property $|a_n| \leq \frac{M}{r^n}$ for any $r > 0$

In particular, if $n < 0$, then

$$\frac{M}{r^n} \to 0 \text{ as } r \to 0$$

and hence we obtain

$$a_n = 0 \quad \forall\ n < 0$$

So the laurent series is in fact a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

which is analytic in the disc $|z - z_0| < \rho$. Thus f extends to an analytic function on $|z - z_0| < \rho$ and so z_0 is a removable singularity of f.