MA 205 Complex Analysis. Dated 28/05/09

Lecture 17

Isolated singularity of f:

a point $z_0 \in \mathbb{C}$ such that f is analytic in $0<|z-z_0|<r$ for some $r>0$

eg. $z=0$ for $f(z) = 1/z$, $\sin z/z$

$z=0$ is not an isolated singularity for $f(z) = \log z$

Removable singularity:

an isolated singularity of f such that f extends to an analytic function on $|z-z_0|<r$ for some $r>0$

Riemann's Theorem:

An isolated singularity z_0 is a removable singularity of $f \iff f$ is bounded on a punctured disc $0<|z-z_0|<r$ for some $r>0$

Laurent decomposition:

f is analytic on the annulus $\rho<|z-z_0|<\sigma$ $(0 \le \rho < \sigma)$

$\Rightarrow f = f_0(z) + f_1(z)$ on $\rho<|z-z_0|<\sigma$ $\forall z \in \mathbb{C}$

where f_0 is analytic on $|z-z_0|<\sigma$

f_1 is analytic on $|z-z_0|>\rho$ and at ∞

and $f_1(\infty)=0$. Moreover f_0 and f_1 are unique.

Laurent series expansion

f analytic on $\rho<|z-z_0|<\sigma$ $(0 \le \rho < \sigma)$

$$\Rightarrow f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

$$= \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

where $a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(\omega)\, d\omega}{(\omega - z_0)^{n+1}}$ for any $r \in \mathbb{R}$ with $\rho < r < \sigma$

In particular, if

$$|f(\omega)| \leq M \quad \forall \ \omega \text{ on } C_r$$

then $$|a_n| \leq \frac{M}{r^n}$$

Suppose f has an isolated singularity at z_0. Then we have a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

Three cases are possible

(i) $a_n = 0 \quad \forall \ n < 0$

In this case $z = z_0$ is a removable singularity of f

(ii) $a_n = 0$ for all except finitely many negetive values of n, & $a_n \neq 0$ for atleast one $n<0$ i.e. $a_{-N} \neq 0$ ~~i.e.~~ & $a_n = 0 \quad \forall \ n < -N$ where $N \in \mathbb{N}$

In this case Laurent series expansion looks like

$$\frac{a_{-N}}{(z-z_0)^N} + \frac{a_{-(N-1)}}{(z-z_0)^{N-1}} + \text{------} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2$$

In this case we say that $z = z_0$ is a pole of order N

Observe that z is a pole of order N.

$$f(z) = \frac{h(z)}{(z-z_0)^N} \qquad \text{for } 0 < |z-z_0| < r \text{ for some } r > 0$$

where $h(z)$ analytic on $|z-z_0| < r$ and $h(z_0) \neq 0$

(iii) $a_n \neq 0$ for infinitely many negetive values of n. In this case we say that f has an essential singularity at z_0.

eg. $\frac{\sin z}{z}$ has a removable singularity at $z=0$ and the laurent series expansion around 0 is

$$f(z) = \frac{1}{z}\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} - - - \right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + - -$$

(ii) $f(z) = \frac{z^2}{(z-1)(z-2)}$.

We have $f(z) = \frac{h_1(z)}{(z-1)}$ on $|z-1| < \frac{1}{2}$

and $f(z) = \frac{h_2(z)}{(z-2)}$ on $|z-2| < \frac{1}{2}$

where $h_1(z) = \frac{z^2}{z-2}$ analytic on $|z-1| < \frac{1}{2}$

$h_2 = \frac{z^2}{z-1}$ analytic on $|z-2| < \frac{1}{2}$

Thus. f has a pole of order at $z=1$ and at $z=2$.

Laurent series expansion

$$\frac{(z-1+1)^2}{(z-1)(z-1-1)} = \frac{-\{(z-1)^2 + 2(z-1) + 1\}}{(z-1)} \sum_{n=0}^{\infty} (z-1)^n$$

$$= -(z-1)\sum_{n=0}^{\infty}(z-1)^n - 2\sum_{n=0}^{\infty}(z-1)^n \div \frac{1}{(z-1)} \sum_{n=0}^{\infty}(z-1)^n.$$

$$= \frac{-1}{z-1} - 3 - 4\sum_{n=1}^{\infty}(z-1)^n$$

exercise :

Calculate the laurent series expansion around pt. $z=2$

(iii) $f(z) = e^{-1/z^2}$

$$f(z) = 1 + \frac{(-1/z^2)}{1!} + \frac{(-1/z^2)^2}{2!} + \cdots$$

$$= 1 - \frac{1}{1!z^2} + \frac{1}{2!z^4} - \frac{1}{3!z^6} + \cdots$$

This is valid for $z \neq 0$. thus $z=0$ is an essential singularity

A way to check if $(z-z_0)$ is a removable singularity

z_0 is a removable singularity

$\Leftrightarrow \quad \lim_{z \to z_0} (z-z_0) f(z) = 0.$

A way to check if $z=z_0$ is a pole of order N

$\lim_{z \to z_0} (z-z_0)^N f(z)$ exist and $\neq 0$.

Def

If f has a pole of order N at $z=z_0$ then

$$f(z) = \frac{a_{-N}}{(z-z_0)^N} + \frac{a_{-N+1}}{(z-z_0)^{-N+1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$$

for $0<|z-z_0|<\sigma$

for some $\sigma>0$, where $a_n \in \mathbb{C}$ with $a_{-N} \neq 0$ are uniquely determined by f. In this case

$$P(z) = \frac{a_{-N}}{(z-z_0)^N} + \frac{a_{-N+1}}{(z-z_0)^{-N+1}} + \cdots + \frac{a_{-1}}{(z-z_0)}.$$

is called the principle part of $f(z)$ at z_0.

Def

A complex valued function f is called meromorphic function if ~~the~~ it is analytic at all points of $\Omega \setminus S = \{z \in \Omega : z \in S\}$ where Ω is an open, connected subset of $\mathbb{C}$ and S is a subset of Ω such that f has an isolated singularity at ~~eq~~ each point of S and moreover, every isolated singularity is a pole of f.

eg. a rational function.

$$f(z) = \frac{p(z)}{q(z)}.$$

where $p(z)$ and $q(z)$ are polynomials and $q(z) \neq 0$, is a meromorphic function on $\mathbb{C}$

Here $\Omega = \mathbb{C}$ and

$$S = \{z_0 \in \mathbb{C} : q(z) = 0\}$$

is a finite set. Infact for every $z_0 \in S$ we can write

$$p(z) = (z-z_0)^r p_1(z). \quad \text{with } p_1(z_0) \neq 0$$
$$q(z) = (z-z_0)^s q_1(z) \quad\quad q_1(z_0) \neq 0$$

and thus

$z = z_0$ is removable singularity if $r \geq s$.

and pole of order $s-r$ if $r < s$.

Residues.

Suppose f has isolated singularity at z_0

Then

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

~~for some $\sigma > 0$~~

for some $0 < |z-z_0| < \sigma$; $\sigma > 0$

where $$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(\omega)\, d\omega}{(\omega - z_0)^{n+1}}$$

C_r is the circle $|z-z_0| = r$. or any simple ~~In particular~~ closed piecewise smooth curve enclosing z_0 and inside $|z-z_0| < \sigma$ | ~~$|z-z_0| < r < \sigma$~~

In particular

$$a_{-1} = \frac{1}{2\pi i} \int f(\omega)\, d\omega$$

This is called the residue of f at $z = z_0$ and denoted by $\operatorname{Res}_{z_0} f(z)$.

eg. $$\int_{|z| = 1/2} \frac{(z-1)^2 \sin z}{z^3}\, dz$$

Here the integrand is

$$\frac{(z^2 - 2z + 1)}{z^3} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right]$$

co-eff of $\frac{1}{z}$ is $= -2$.

$$\operatorname{Res}_{z_0} f(z) = -2.$$

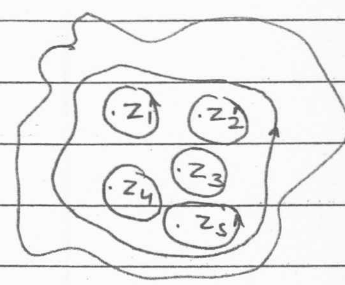
$$I = 2\pi i(-2)$$
$$= -4\pi i.$$

Cauchy Residue Theorem :

Suppose f is analytic on an open connected set $\Omega \subseteq \mathbb{C}$ except at finitely many points $z_1, z_2, z_3 \ldots z_r$ of Ω. Then each z_i is an isolated singularity of f and if C is a simple closed (piecewise smooth curve in Ω enclosing $z_1, z_2 \ldots z_r$, then

$$\int_C f(z)\,dz = 2\pi i \sum_{i=1}^{r} \operatorname{Res}_{z_i} f(z).$$

proof



Since $z_1, z_2 \ldots z_r$ are finite in number, we can find discs $D_1, D_2 \ldots D_r$ that are disjoint and centered at $z_1, z_2 \ldots z_r$ respectively

Thus $z_1, z_2 \ldots z_r$ are isolated singularities of f

Let $C_i = \partial D_i$

Now $C - C_1 - C_2 - C_3 \ldots - C_r$ enclose a region where f is analytic.

By Cauchy's theorem

$$\int_{C - C_1 - C_2 - \cdots - C_r} f(z)\,dz = 0$$

$$\Rightarrow \sum_{i=1}^{r} \int_{C_i} f(z)\,dz \Rightarrow \cancel{\frac{1}{2\pi i}} \sum_{i=1}^{r} 2\pi i \left(\operatorname{Res}_{z_i} f\right) \text{ as desired.}$$