MA 205 Complex Analysis.

Lecture 18 Dated 29/05/09

isolated singularity.

removable singularity

$\lim_{z \to z_0} (z-z_0) f(z) = 0$

pole (of order N)

$f(z) = \dfrac{h(z)}{(z-z_0)^N}$

for some $N>0$

$h(z)$ analytic around z_0 & $h(z_0) \neq 0$

essential singularity

Laurent series at z_0 has infinitely many terms with negetive exponents

These concepts also extend to the case when $z_0 = \infty$ by considering $g(w) = 1/f(w)$ for $w \neq 0$. Now f has an isolated singularity (respectively removable, pole and essential singularity) at ∞ if g has it a 0

A meromorphic function on an open (connected) set $\Omega \subseteq \hat{\mathbb{C}}$ is a function f on Ω except at some isolated singularities all of which are poles.

eg. A rational function is a meromorphic function

Indeed if $f(z) = \dfrac{p(z)}{q(z)}$ where $p(z), q(z)$ are polynomials with non-constant common factor

then zeros of $f(z)$ in $\mathbb{C}$ $\longleftrightarrow$ zeros of $p(z)$

poles of $f(z)$ in $\mathbb{C}$ $\longleftrightarrow$ zeros of $q(z)$.

At $z=\infty$, if

$$p(z) = a_m z^m + \cdots + a_1 z + a_0$$
$$q(z) = b_n z^n + \cdots + b_1 z + b_0$$

where $a_i, b_j \in \mathbb{C}$ with $a_m \neq 0$, $b_n \neq 0$

then $f(z)$ has $\begin{cases} \text{a zero } \cancel{\text{of}} \text{ at } \infty \text{ of order } n-m & \text{if } m<n \\ \text{a pole at } \infty \text{ of order } m-n & \text{if } m>n \\ \text{analytic at } \infty \text{ with } f(\infty) = \frac{a_m}{b_n} \neq 0 & \text{if } m=n \end{cases}$

Observe that as a consequence,
No. of zeros of $f(z)$ in $\hat{\mathbb{C}}$, counting multiplicities
= number of poles of $f(z)$ in $\hat{\mathbb{C}}$, counting multiplicities

Remark :

A meromorphic function on $\hat{\mathbb{C}}$ is necessay a rational function.

eg. $f(z) = e^{-1/z}$
$$= 1 - \frac{1}{z} + \frac{1}{2!z^2} - \frac{1}{3!z^3} + \cdots$$

has an essential singularity at 0

If $z = x \in \mathbb{R}$

then $f(x) = 0$ as $x \to 0^+$

and $f(x) = \infty$ as $x \to 0^-$

In general if $z = x+iy$

$$\Rightarrow \quad \frac{-1}{z} = \frac{-(x-iy)}{x^2+y^2}$$

and $e^{-1/z} = e^{-x/x^2+y^2} e^{iy/x^2+y^2} = re^{i\theta}$

where $r = e^{-x/x^2+y^2}$ $\quad \theta = y/x^2+y^2$

One can see that

$f(z)$ takes all possible value in $\mathbb{C}$ except $w=0$; also there is a sequence (z_n) such that $z_n \to 0$ & $f_n(z) \to 0$

Theorem :

[Casorati - Weierstrass theorem) :

If f has an essential singularity at $z_0 \in \mathbb{C}$ then for any $w_0 \in \mathbb{C}$, there is sequence (z_n) of point in the domain of f such that $z_n \to z_0$ and $f(z_n) \longrightarrow w_0$.

proof : let w_0 be any complex value

suppose, if possible, there is no sequence (z_n) such that $z_n \to z_0$ and $f(z_n) \to w_0$

Then there is a $\varepsilon > 0$ such that

$$|f(z) - w_0| \geq \varepsilon \qquad \text{for all } z \text{ near } z_0$$

Now consider,

$$h(z) = \frac{1}{f(z) - w_0}$$

Then h is defined in a punctured disk $0<|z-z_0|<r$ for some $r>0$ and is analytic (since z_0 was an isolated singularity.) . Moreover,

$$|h(z)| \leq \frac{1}{\varepsilon} \qquad \text{for } 0<|z-z_0|<r$$

and so h is bounded near z_0

So by Riemann's theorem, z_0 is a removable singularity of h.

Thus

$$h(z) = (z-z_0)^N g(z)$$

for some $N \geq 0$ and $g(z)$ analytic in $|z-z_0|<r$ with $g(z_0) \neq 0$.

hence

$$f(z) - \omega_0 = (z-z_0)^{-N} \frac{1}{g(z)}$$

$\Rightarrow f(z) = \omega_0 + \frac{(z-z_0)^{-N}}{g(z)}$ where $1/g(z)$ is analytic at z_0

$$= \omega_0 + \frac{1/g(z)}{(z-z_0)^N}$$

$$= \frac{\omega_0 (z-z_0)^N + 1/g(z)}{(z-z_0)^N}$$

$\Rightarrow f(z)$ has a pole at z_0 or a removable singularity at z_0. which is a contradiction.

Remark:

One has the following two nice theorems of Picard which we'll not proove.

Picard's Little theorem

A non constant entire function misses at most one value in $\mathbb{C}$.

Picard's Big Theorem:

If f has an essential singularity ~~to~~ at z_0, then f attains every value in $\mathbb{C}$ with atmost one exception, infinitely often near z_0. In other words, $\exists\, \omega^* \in \mathbb{C}$ such that $\forall\, \omega_0 \in \mathbb{C} \setminus \{\omega^*\}$, there is a sequence (z_n) of distinct points such that $z_n \to z_0$ & $f(z_n) = \omega_0 \;\; \forall n$.

Cauchy Residue Theorem :

If C is a simple closed curve and f is analytic on and inside C except at finitely many points $z_1, z_2 \,---\, z_n$ then

$$\int_C f(z)\,dz = 2\pi i \sum_{i=1}^{n} \operatorname{Res}_{z_i} f(z)$$

eg. $\int f(z)\,dz = 2\pi i \sum\left[\operatorname{Res}_{z=1} f + \operatorname{Res}_{z=2i} f + \operatorname{Res}_{z=-2i} f\right]$

where

$$f(z) = \frac{1}{(z^2+4)(z-1)}$$ so $f(z)$ has simple pole

$$\operatorname{Res}_{z=1} f(z) = \lim_{z\to 1} (z-1) f(z) = \frac{1}{5}$$

$$\operatorname{Res}_{z=2i} f(z) = \lim_{z\to 2i} (z-2i) f(z) = \lim_{z\to 2i} \frac{1}{(z-1)(z+2i)}$$

$$= \frac{1}{(2i-1)(4i)} = \frac{-1-2i}{5(4i)}.$$

$$= \frac{-2+i}{20}.$$

$$\operatorname{Res}_{z=-2i} f(z) = \lim_{z\to -2i} \frac{1}{(z-2i)(z-1)} = \frac{1}{(-4i)(-2i-1)} = \frac{1}{4i(1+2i)}.$$

$$= \frac{-i(1-2i)}{4(1+4)} = \frac{-i-2}{20} = \frac{-2-i}{20}.$$

$$\int_{C_r} f(z)\,dz = 2\pi i\left[\frac{4}{20} + \frac{\cancel{i}-2}{20} + \frac{-2-\cancel{i}}{20}\right]$$

$$= 2\pi i\left[\frac{0}{20}\right] = 0.$$

How to calculate residues.

Rule 1

If f has a simple pole at z_0, then

$$\operatorname{Res}_{z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z)$$

Rule 2

If f has a double pole at z_0 then

$$\operatorname{Res}_{z_0} f(z) = \lim_{z \to z_0} \frac{d}{dz} \left[(z - z_0)^2 f(z) \right]$$

Rule 3 (Special case of Rule 1) : If $f(z)$ and $g(z)$ are analytic at z_0 and $g(z)$ has a simple pole at z_0, then

$$\operatorname{Res}_{z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}$$

eg. $\operatorname{Res}_{i} \frac{z^3}{z^2+1} = \lim_{z \to i} \frac{z^3}{2z} = \frac{-1}{2}$

Rule 4 (Special case of Rule 3) : If $g(z)$ is analytic & has a simple zero at z_0, then

$$\operatorname{Res}_{z_0} \left(1/g(z) \right) = \frac{1}{g'(z_0)}$$

Improper Integrals.

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx.$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{S\to -\infty} \int_S^b f(x)\,dx.$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$

provided both the integrals on RHS exist
In this case 0 can be replaced by any $a \in \mathbb{R}$

If $\int_{-\infty}^\infty f(x)\,dx$ is convergent i.e. if both $\lim_{R\to\infty}\int_0^R f(x)\,dx$ and $\lim_{S\to -\infty}\int_S^0 f(x)\,dx$ exist

then $\int_{-\infty}^\infty f(x)\,dx = \lim_{R\to\infty}\int_{-R}^R f(x)\,dx$

However converse is not true, eg. if $f(x) = x$

then

$$\int_{-R}^R f(x)\,dx = 0$$

and so

$$\lim_{R\to\infty}\int_{-R}^R f(x)\,dx = 0$$

but neither $\int_0^\infty f(x)\,dx$ nor $\int_{-\infty}^0 f(x)\,dx$ exist.

In general

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

when it exist, is called the cauchy principal value of

$$\int_{-\infty}^{\infty} f(x)\,dx$$

If f is an even function, then $\int_{-R}^{R} f(x)\,dx$ $= 2\int_{0}^{R} f(x)\,dx$. and in this case

Cauchy Principle value (P.V.) exist $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx$ is convergent and equal to (P.V.)

eg.

$$\int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}\,dx$$

here $f(x) = \dfrac{2x^2-1}{x^4+5x^2+4}$. is an even function

and so it is enough to find

$$\int_{-R}^{R} f(x)\,dx$$ & its limit as $R\to\infty$

Consider corresponding complex function

$$f(z) = \frac{2z^2+1}{z^4+5z^2+4} = \frac{2z^2-1}{(z^2+4)(z^2+1)}$$

singularities at $z = \pm i, \pm 2i$ of order 1 and a semi-circle C_r joining $(R,0)$ to $(-R,0)$

Clearly $\displaystyle\int_{\gamma_R} f(z)\,dz. = \int_{-R}^{R} f(x)\,dx + \int_{C_r} f(z)\,dz.$

By residue theorem.

LHS $\Rightarrow 2\pi i \left[\operatorname{Res}_{i} f(z) + \operatorname{Res}_{2i} f(z) \right]$

$$= 2\pi i \left[\frac{2i^2-1}{4i^3+10i} + \frac{2(2i)^2-1}{4(2i)^3+10(2i)} \right]$$

$$= 2\pi i \left[\frac{-2-1}{-4i+10i} + \frac{-8-1}{-32i+20i} \right].$$

$$= 2\pi i \left[\frac{-\cancel{3}\,1}{\cancel{6}i\,2} + \frac{-\cancel{9}\,3}{-\cancel{12}i\,4} \right]$$

$$= 2\pi i \left[\frac{-1}{2i} + \frac{3}{4i} \right].$$

$$= \cancel{2}\pi \cancel{i} \left[\frac{1}{\cancel{4}\cancel{i}} \right] = \frac{\pi}{2}$$

on C_r we have $|z| = R$ and

$$|2z^2-1| \leq 2R^2+1$$

and $|(z^2+1)(z^2+4)| \geq (|z|^2-1)\,(|z|^2-4).$

$$f(z) \leq \frac{2R^2+1}{(R^2-1)(R^2-4)}.$$

hence $$\left| \int f(z)\,dz \right| \leq \frac{2R^2+1}{(R^2-1)(R^2-4)} \cdot \pi R.$$

$$\to 0 \quad \text{as } R \to \infty$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{2}$$