$\Rightarrow |z_n| \leq |z|+1 \quad \forall \; n \geq n_0$

$\Rightarrow |z_n| \leq |z|+1 \; \forall \; n \geq n_0$

So, if $M = \max \{ |z|+1, |z_1|, |z_2|, \ldots, |z_{n-1}| \}$

then $|z_n| \leq M \; \forall \; n \in \mathbb{N}$

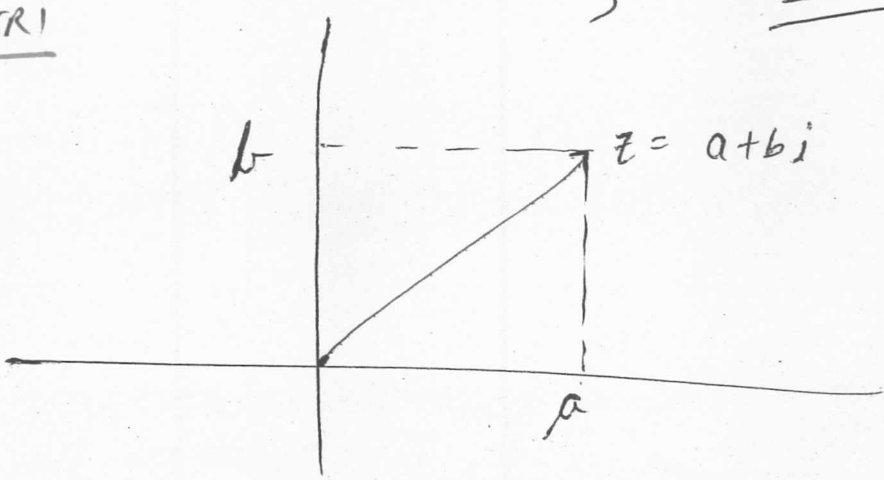
Cauchy Sequence → works for $\mathbb{C}$

Monotonic Sequence → do not works for $\mathbb{C}$

Lecture 2

Notes by KSHITIZ KHATRI

$\mathbb{C} = \{ a+bi \; : \; a, b \in \mathbb{R} \}$. 7/5/09

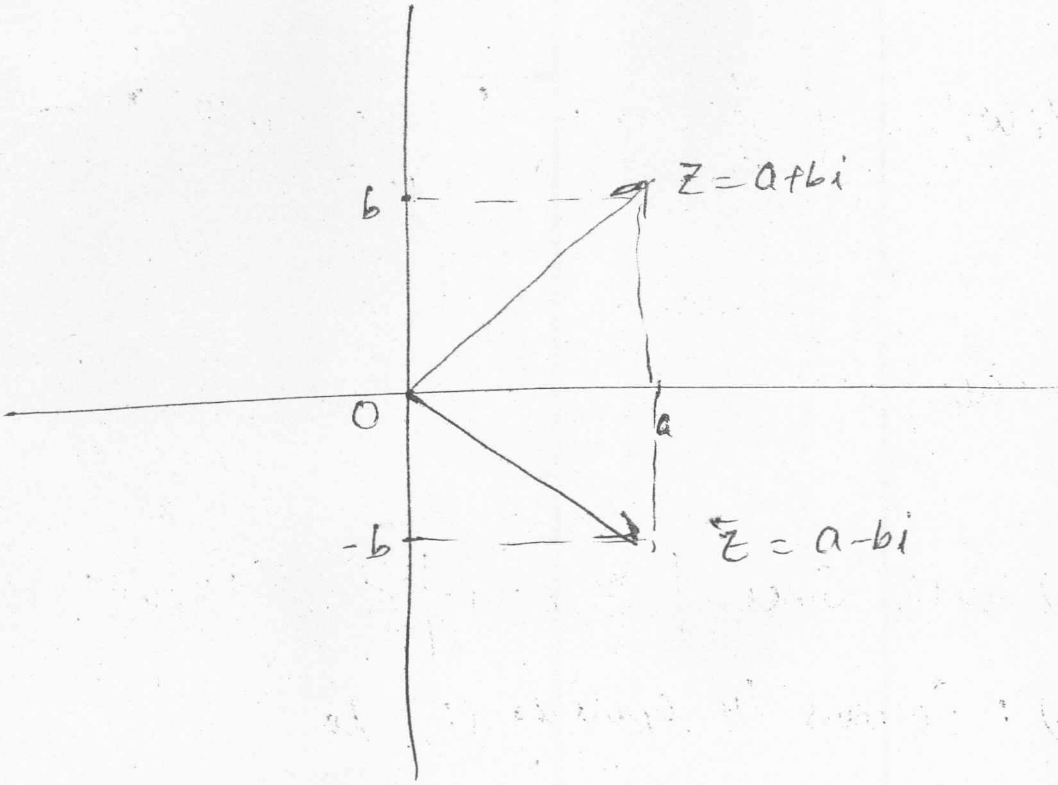


For $z = a+bi \in \mathbb{C}$,

$|z| = \sqrt{a^2+b^2}$

P.T.O.

$\bar{z} = a - bi$



We showed

$$\overline{z+w} = \bar{z} + \bar{w} \quad ; \quad \overline{zw} = \bar{z}\,\bar{w}$$

$$|z+w| \le |z| + |w|$$

$$\big|\,|z| - |w|\,\big| \le |z - w|$$

<u>Cauchy - Schwartz Inequality</u>

For $z_1, z_2, \ldots z_n, w_1, w_2, \ldots \ldots w_n \in \mathbb{C}$

$$\left| \sum_{i=1}^{n} z_i w_i \right| \le \left(\sum_{i=1}^{n} |z_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |w_i|^2 \right)^{1/2} \text{ — } (1)$$

(Other forms of Cauchy-Schwartz Inequality)

$$\sum_{i=1}^{n} |z_i w_i| \le \left(\sum_{i=1}^{n} |z_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |w_i|^2 \right)^{1/2} \quad (2)$$

$$\left| \sum_{i=1}^{n} z_i \bar{w}_i \right| \le \quad (3)$$

Note: (2) $\Rightarrow$ (1) Since $\left| \sum z_i w_i \right| \le \sum |z_i w_i|$

(1) $\Rightarrow$ (2) : Follows by applying (1) to

$|z_1|, |z_2|, |z_3| - - -$

Another way to view C-S Inequality is by defining an "inner product" or "dot product on

$$\mathbb{C}^n = \{ \underline{z} = (z_1, \ldots, z_n) : z_1, \ldots z_n \in \mathbb{C} \}$$

by

$$\langle \underline{z}, \underline{w} \rangle = \underline{z} \cdot \underline{w} = z_1 \bar{w}_1 + z_2 \bar{w}_2 + \cdots + z_n \bar{w}_n$$

i) $\langle \underline{z} + \underline{z}', w \rangle = \langle \underline{z}, w \rangle + \langle \underline{z}', w \rangle$ &

$\langle \underline{z}, \underline{w} + \underline{w}' \rangle = \langle \underline{z}, \underline{w} \rangle + \langle \underline{z} + \underline{w}$

$$\langle k\underline{z}, w\rangle = k\,\langle \underline{z}, \underline{w}\rangle \;\cancel{\ne \langle \underline{z}, k\underline{w}\rangle}$$

$$\& \quad \langle \underline{z}, k\underline{w}\rangle = \bar{k}\,\langle \underline{z}, \underline{w}\rangle \;\; \forall k \in \mathbb{C}$$

$$\langle \underline{z}, \underline{w}\rangle = \overline{\langle \underline{w}, \underline{z}\rangle}$$

$$\langle \underline{z}, \underline{z}\rangle \ge 0 \quad \text{and}$$

$$\langle \underline{z}, \underline{z}\rangle = 0 \iff \underline{z} = \underline{0}$$

One defines the <u>norm</u> of $\underline{z} = (z_1, z_2, \ldots z_n) \in \mathbb{C}^n$

$$\|\underline{z}\| = \langle \underline{z}, \underline{z}\rangle^{1/2} \quad (\text{"length of } \underline{z} \text{ from } \underline{0}).$$

We have

$$\|\underline{z}\| \ge 0 \quad \text{and} \quad \|\underline{z}\| = 0 \iff \underline{z} = 0$$

$$\|k\underline{z}\| = |k|\,\|\underline{z}\| \quad \forall\, k \in \mathbb{C} \;\&\; \underline{z} \in \mathbb{C}^n$$

In this notation, we can write (3) as:

$$(*) \quad |\langle \underline{z}, \underline{w}\rangle| \le \|\underline{z}\|\,\|\underline{w}\| \quad \forall\, \underline{z}, \underline{w} \in \mathbb{C}^n$$

Proof: Consider a real variable t, and look at

$$0 \le \langle t\underline{z} + \underline{w}, t\underline{z} + \underline{w}\rangle = \boxed{t^2\langle \underline{z}, \underline{z}\rangle + 2t\,\mathrm{Re}\{\langle \underline{z}, \underline{w}\rangle\} + \langle \underline{w}, \underline{w}\rangle}$$

~~Hence, for the quadratic equation formed we can say~~

$$b^2 - 4ac \leq 0 \quad \text{where } b = 2\mathrm{Re}\langle \underline{z}, \underline{w} \rangle, \ \forall t \in \mathbb{R}$$

$$a = \langle \underline{z}, \underline{z} \rangle \ \& \ c = \langle \underline{w}, \underline{w} \rangle$$

~~i.e. [2 Re~~

<u>Proof</u>: Consider a real variable t, and look at

$x_i = |z_i|$, $y_i = |w_i|$, we'll prove (2)

and

$$0 \leq \langle t\underline{x} + \underline{y}, t\underline{x} + \underline{y} \rangle =$$

$$t^2 \|x\|^2 + 2t \langle \underline{x}, \underline{y} \rangle + \|y\|^2$$

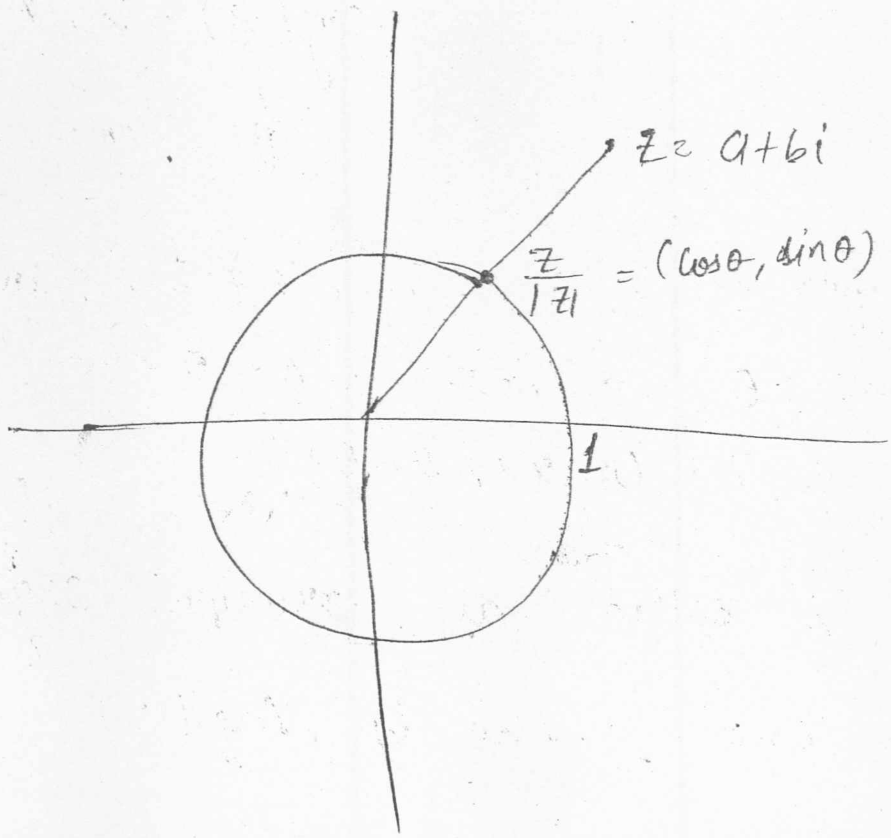
so

$$b^2 - 4ac \leq 0 \implies \langle \underline{x}, \underline{y} \rangle^2 \leq \|x\|^2 \|y\|^2$$

$$\implies |\langle \underline{x}, \underline{y} \rangle| \leq \|\underline{x}\| \|\underline{y}\|$$

This proves (2) & hence (1).

Arguement of a Complex Number →



If $z = a+bi \in \mathbb{C}$ and $z = 0$, then any $\theta \in \mathbb{R}$ such that

$$z = |z|\left[\cos\theta + i\sin\theta\right]$$

is called an <u>arguement</u> of z and denoted by $\arg(z)$

→ Note that for any $z \neq 0$,

$\arg(z)$ exists and is unique upto the addition of an integer multiple of 2π.

Further, for $z, \omega \in \mathbb{C} \setminus \{0\}$,

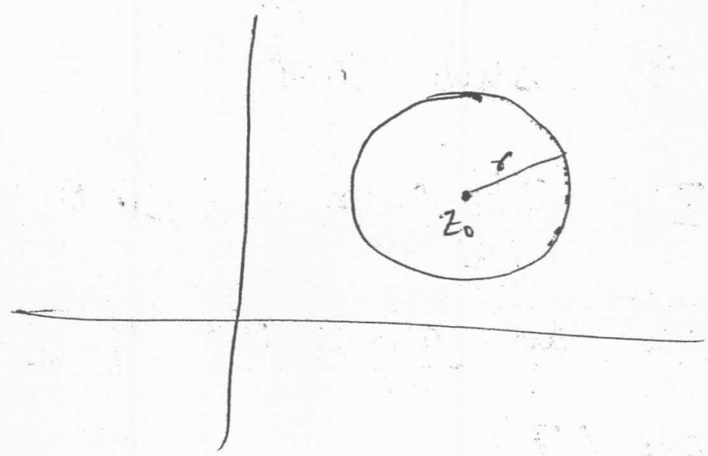
P.T.O.

$$z = w \iff |z| = |w| \text{ and } \arg(z) = \arg(w) + 2n\pi$$
for some $n \in \mathbb{Z}$

Further for any $z \in \mathbb{C}$, $z \neq 0$, there is a unique $\theta \in \mathbb{R}$ s.t. $-\pi < \theta < \pi$ and
$$z = |z|(\cos\theta + i\sin\theta).$$

This θ is called as <u>principal argument</u> of z and denoted by $\operatorname{Arg}(z)$

<u>Definition:</u>



Given any $z_0 \in \mathbb{C}$ and $r \in \mathbb{R}$ with $r \geq 0$, we define

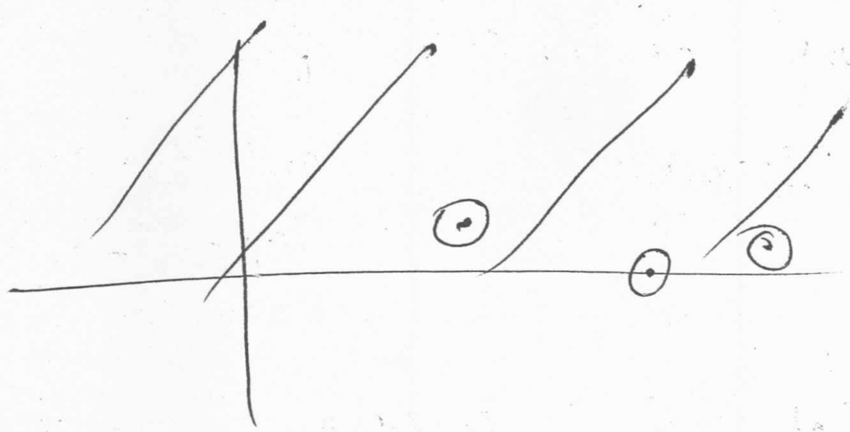
$$B(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}$$

This is called the open ball or open disk centered at z_0 of radius r.

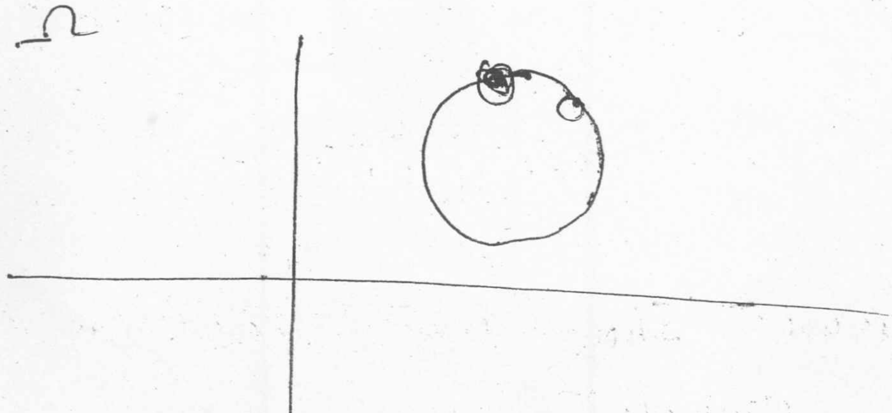
<u>Definition:</u> If $\Omega \subseteq \mathbb{C}$, then a point z_0 of Ω is called an <u>interior point</u> of Ω if $B(z_0, r) \subseteq \Omega$ for some $r > 0$

<u>Ex:</u> ① If $\Omega = \{z \in \mathbb{C} : Im(z) \geq 0\}$ then the interior points of Ω are those $z \in \mathbb{C}$ for which $Im(z) > 0$

i.e.



② If $\Omega = B(w, r)$ for some $w \in \mathbb{C}$ & $r > 0$ then every point of Ω is an interior pt. of Ω



P.T.O.

Def: A subset Ω of $\mathbb{C}$ is said to be open if every pt. of Ω is an interior pt. of Ω.

Continuity $\rightarrow$

Let $\Omega \subseteq \mathbb{C}$ and $f: \Omega \rightarrow \mathbb{C}$ be any function. We say that f is continuous at $z_0 \in \Omega$ if

(z_n) sequence in Ω and $z_n \rightarrow z_0 \Rightarrow$
$$f(z_n) \rightarrow f(z_0)$$

Equivalent Def: f is continuous at z_0 if for every $\epsilon > 0$,

$\exists$ $\delta > 0$ s.t.

$$z \in \Omega \ \& \ |z - z_0| < \delta$$
$$\Rightarrow |f(z) - f(z_0)| < \epsilon$$

Exercise: Show that these two definitions are equivalent

Examples

1) $f(z) = z$ for $z \in \Omega = \mathbb{C}$.

f is cont. at every $z_0 \in \mathbb{C}$

[poly. f^n]

(2) $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0$

for $z \in \mathbb{C}$

f is continuous at every $z_0 \in \mathbb{C}$

3) $f(z) = \frac{1}{z}$ for $z \in \Omega = \mathbb{C} \setminus \{0\}$

More generally, if f is a rational f^n

i.e. $f(z) = \dfrac{p(z)}{q(z)}$, $z \in \Omega$

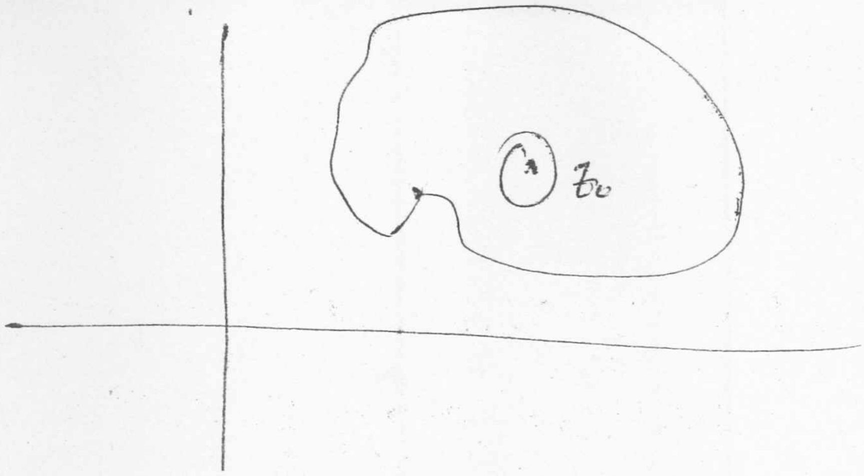
where $p(z)$, $q(z)$ and poly fns. and Ω doesn't contain any roots of $q(z)$, then f is continuous on Ω.

__Def:__ Let $\Omega \subseteq \mathbb{C}$ and $f: \Omega \to \mathbb{C}$ be any function.

Also let z_0 be an interior point of Ω.

We say that f is __differentiable__ at z_0

if $\lim\limits_{h \to 0} \dfrac{f(z_0+h) - f(z_0)}{h}$ $(h \in \mathbb{C})$ exist



In other words, if $\exists\, l \in \mathbb{C}$ s.t.

$\forall\, \epsilon > 0$, $\exists\, \delta > 0$ s.t.

$h \in \mathbb{C}$ & $0 < |h| < \delta \Rightarrow$

$$\left| \frac{f(z_0+h) - f(z_0)}{h} - l \right| < \epsilon$$

Suppose we write

$$z = x + iy$$

and

$$f(z) = u + iv$$

then u, v can be considered as real-valued functions of two variables x, y

$u(x,y)$, $v(x,y)$

Suppose f is diff. at $z_0 = x_0 + iy_0$ and

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} = l = l_1 + i l_2$$

$$\& \; h = h_1 + i h_2$$

then ~~we have~~ letting $h \to 0$ along the x-axis,

i.e. consider $h = h_1 + 0.i$ $(h_1 \in \mathbb{R})$

then

~~$u(x_0+h_1, y_0) - u(x_0, y_0) + i[V(x_0+h\ y_0$~~

$$\lim_{h_1 \to 0} \frac{u(x_0+h_1, y_0) - u(x_0, y_0) + i\left[V(x_0+h_1, y_0) - V(x_0, y_0)\right]}{h_1}$$

$$= l = l_1 + il_2$$

This implies

$$\lim_{h_1 \to 0} \frac{u(x_0+h_1, y_0) - u(x_0, y_0)}{h_1} = l_1$$

$$\& \lim_{h_1 \to 0} \frac{V(x_0+h_1, y_0) - V(x_0, y_0)}{h_1} = l_2$$

i.e. $U_x(x_0, y_0)$ exists and is $= l_1$

and $V_x(x_0, y_0)$ exists and is $= l_2$.

Similarly letting $h \to 0$ along the y-axis, i.e. $h = ih_2$, $h_2 \to 0$

we see that

$U_y(x_0, y_0)$ & $V_y(x_0, y_0)$ exists

and $\frac{1}{i}\left[U_y(x_0, y_0) + i V_y(x_0, y_0)\right]$

$= l_1 + il_2$

$\Rightarrow u_y = -l_2$ & $v_y = l_1$

So if f is diff. at $z_0 = x_0 + iy_0$

then u_x, u_y, v_x, v_y exist at (x_0, y_0)

and

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= v_x \end{aligned}} \rightarrow \text{Cauchy - Riemann eq}^n\text{s.}$$